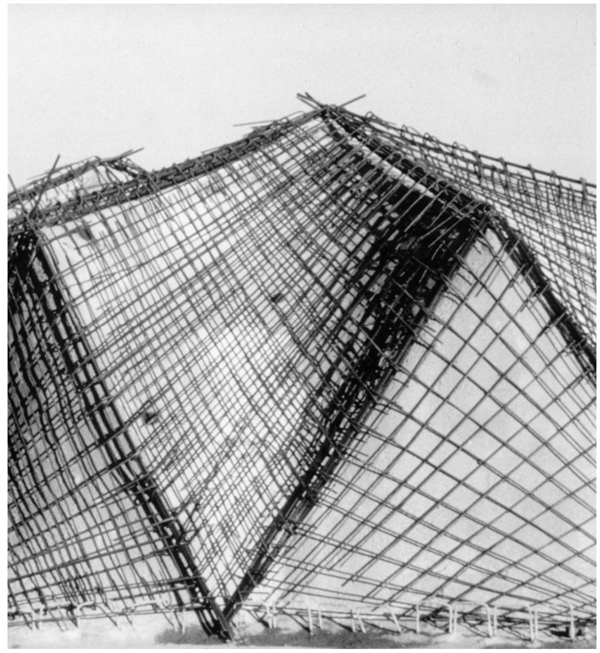
**Architecture 324**
**Structures II**

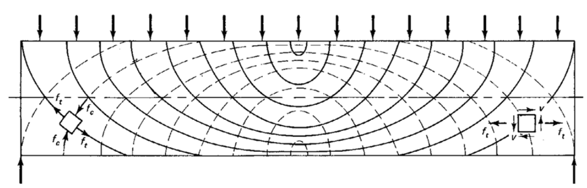
# Reinforced Concrete Beams Ultimate Strength Design
(ACI 318-19) – PART I



- Flexure in Concrete
- Ultimate Strength Design (LRFD)
- Failure Modes
- Flexure Equations
- Rectangular Beam Analysis

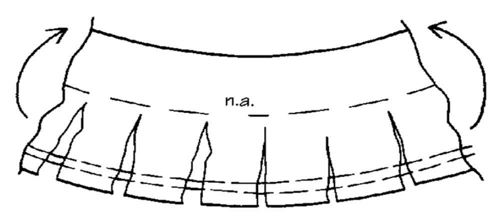


# Flexure



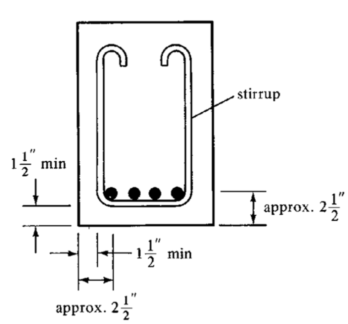
The stress trajectories in this simple beam, show principal tension as solid lines.

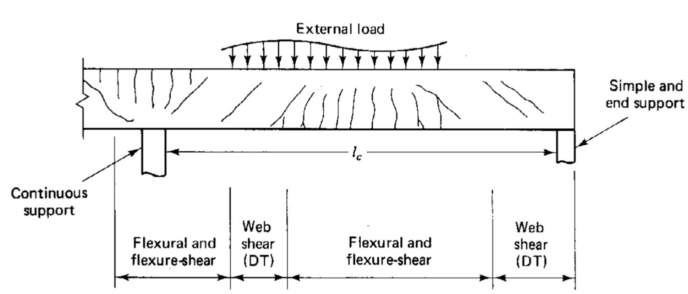
Reinforcement must be placed to resist these tensile forces




In beams continuous over supports, the stress reverses (negative moment).
In such areas, tensile steel is on top.

Shear reinforcement is provided by vertical or sloping stirrups.



# Ultimate Strength – (LRFD)

Nominal Strength ≥ Design Strength
(strength of member ≥ required by loads)

LRFD uses 2 safety factors: $\gamma$ and $\phi$
$\phi$ nominal strength ≥ $\gamma$ required strength

$\gamma$ increases the required strength of the member and is placed on the loads

$\phi$ reduces the member strength capacity and is placed on the calculated force

Loads increased:
$\gamma$ Factors: DL=1.2 LL=1.6
U is the required strength
U=1.2DL+1.6LL
(factors from ASCE 7)

Strength reduced:
$\phi$ Factors: e.g. flexure = 0.9
in tension-controlled beams

**Table 21.2.1—Strength reduction factors $\phi$**

| | Action or structural element | $\phi$ | Exceptions |
|---|---|---|---|
| (a) | Moment, axial force, or combined moment and axial force | 0.65 to 0.90 in accordance with 21.2.2 | Near ends of pretensioned members where strands are not fully developed, $\phi$ shall be in accordance with 21.2.3. |
| (b) | Shear | 0.75 | Additional requirements are given in 21.2.4 for structures designed to resist earthquake effects. |
| (c) | Torsion | 0.75 | — |
| (d) | Bearing | 0.65 | — |
| (e) | Post-tensioned anchorage zones | 0.85 | — |
| (f) | Brackets and corbels | 0.75 | — |
| (g) | Struts, ties, nodal zones, and bearing areas designed in accordance with strut-and-tie method in Chapter 23 | 0.75 | — |
| (h) | Components of connections of precast members controlled by yielding of steel elements in tension | 0.90 | — |
| (i) | Plain concrete elements | 0.60 | — |
| (j) | Anchors in concrete elements | 0.45 to 0.75 in accordance with Chapter 17 | — |



# Ultimate Strength – (ACI 318-14)

Reduced Nominal Strength ≥ Factored Load Effects

$\Phi Sn \geq U$

$\gamma$ **Factored Loads** (see ACSE 7)

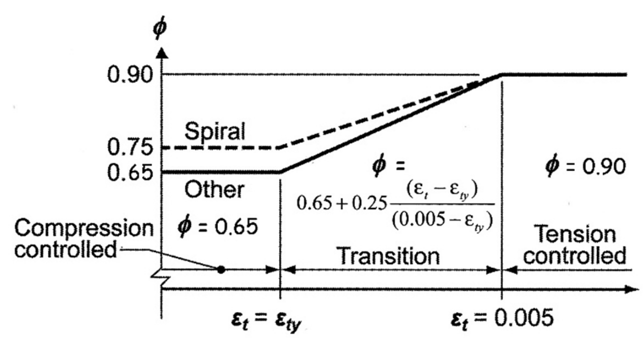
1) 1.4D
2) 1.2D + 1.6L + 0.5(Lr or S or R)
3) 1.2D + 1.6(Lr or S or R) + (1.0L or 0.5W)
4) 1.2D + 1.0W + 1.0L + 0.5(Lr or S or R)
5) 1.2D + 1.0E + 1.0L + 0.2S
6) 0.9D + 1.0W
7) 0.9D + 1.0E

D = service dead loads
L = service live load
Lr = service roof live load
S = snow loads
W = wind loads
R = rainwater loads
E = earthquake loads

**Strength Reduction Factors, $\Phi$**

| | | |
|---|---|---|
| Mn | Flexural ($\varepsilon$ > 0.005) | 0.90 |
| Vn | Shear | 0.75 |
| Pn | Compression (spiral) | 0.75 |
| Pn | Compression (other) | 0.65 |
| Bn | Bearing | 0.65 |
| Tn | Torsion | 0.75 |
| Nn | Tension | 0.90 |
| Combined stress | | 0.65 to 0.90 |

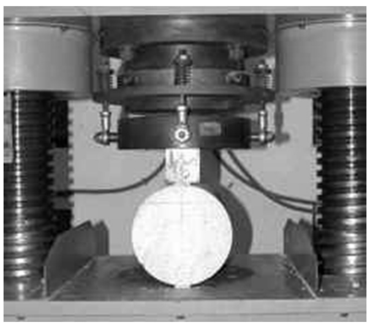
ACI 318 21.2.2

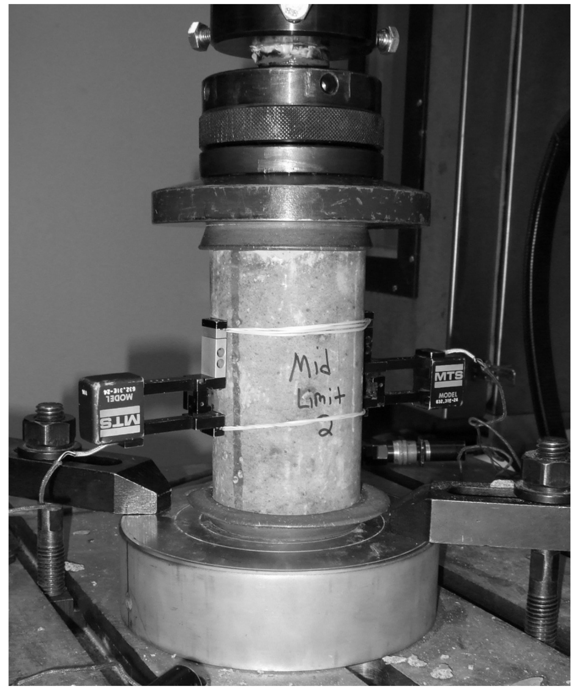


# Strength Measurement

- Compressive strength $f'_c$
  - 12” x 6” cylinder
  - 28 day moist cure
  - Ultimate (failure) strength
  - Usable strain $\varepsilon_{cu}$ = 0.003 (ACI 318)

- Tensile strength ASTM C496 $f'_t$
  - 12” x 6” cylinder
  - 28 day moist cure
  - Ultimate (failure) strength
  - Split cylinder test
  - ca. 10% of f’c
  - Neglected in flexure analysis








# Failure Modes Based on As

$$\rho = \frac{A_s}{bd}$$

- **No Reinforcing**
  - Less than $As_{min}$
  - Brittle failure

As,min:
greater of a and b

$$\text{(a)}\ \frac{3\sqrt{f'_c}}{f_y} b_w d$$

$$\text{(b)}\ \frac{200}{f_y} b_w d$$

- **Reinforcing < balance (use this)**
  - Steel yields before concrete fails
  - Ductile failure
  - ($\sim As_{min}$) 0.06 ≥ $\varepsilon_t$ ≥ 0.004 ($\sim As_{max}$)
  - $\varepsilon_t$ ≥ 0.005 for tension controlled

$$As_{max}\ when\ \boldsymbol{\varepsilon}_t = 0.004$$

- **Reinforcing = balance**
  - Concrete fails just as steel yields
  - $\varepsilon_t$ at balance = 0.0285
    for Gr 60 ksi steel with 4000 psi concrete

$$\rho_{bal} = \left(\frac{0.85\beta_1 f'_c}{f_y}\right)\left(\frac{87000}{87000 + f_y}\right)$$

- **Reinforcing > balance**
  - Concrete fails before steel yields
  - Low ductility
  - Sudden failure

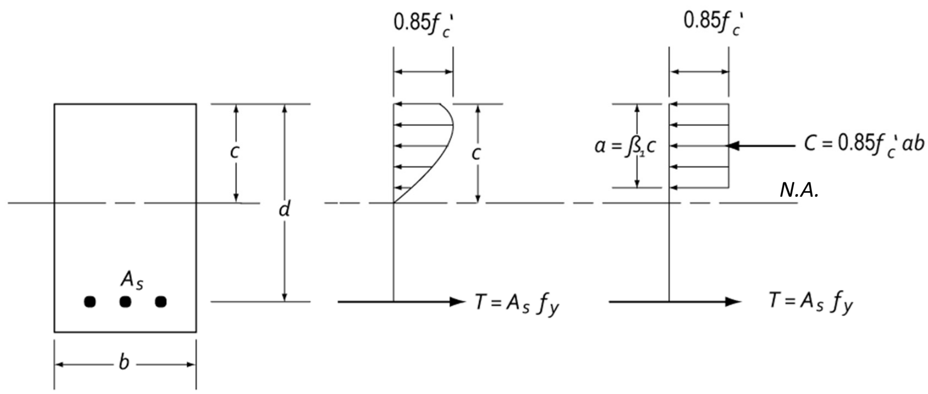
$$As > As_{max} \quad \text{SuddenDeath!!}$$

# ACI Stress Block

$\beta_1$ is a factor to account for the non-linear shape of the compression stress block.

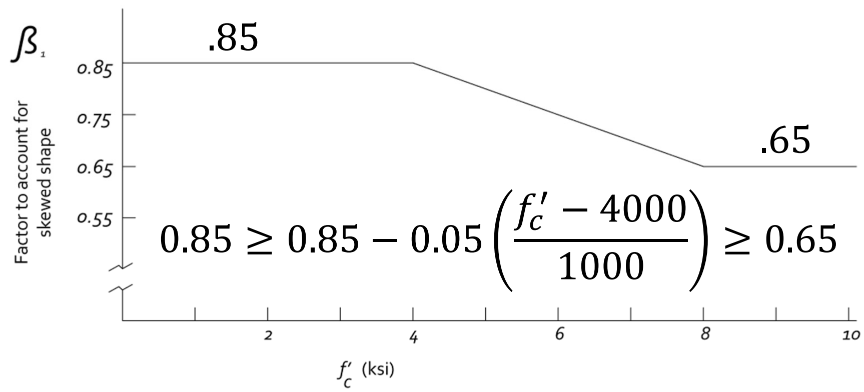
$$a = \beta_1 c$$

Actual stress block — ACI equivalent stress




psi

| f'c | $\beta_1$ |
|---|---|
| 0 | 0.85 |
| 1000 | 0.85 |
| 2000 | 0.85 |
| 3000 | 0.85 |
| 4000 | 0.85 |
| 5000 | 0.8 |
| 6000 | 0.75 |
| 7000 | 0.7 |
| 8000 | 0.65 |
| 9000 | 0.65 |
| 10000 | 0.65 |

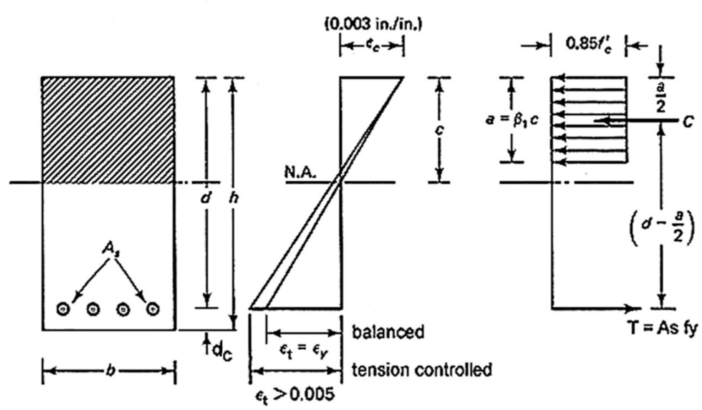



$$0.85 \geq 0.85 - 0.05\left(\frac{f_c' - 4000}{1000}\right) \geq 0.65$$



# Flexure Equations

strain — ACI equivalent stress block




$$C = T$$

$$0.85 f_c' ab = A_s f_y$$

solving for $a$,

$$a = \frac{A_s f_y}{0.85 f_c' b} = \frac{\rho f_y d}{0.85 f_c'}$$

$$\varepsilon_t = \frac{d - c}{c}(0.003)$$

$$\rho = \frac{A_s}{bd}$$

$$M_n = T\left(d - \frac{a}{2}\right) = A_s f_y\left(d - \frac{a}{2}\right)$$

$$M_u = \phi M_n$$

$$M_u = \phi M_n = \phi A_s f_y\left(d - \frac{a}{2}\right)$$

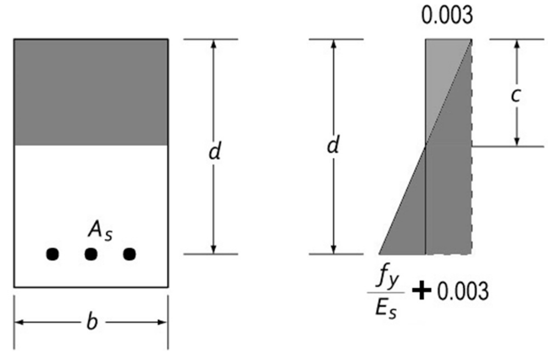
$$M_u = \phi A_s f_y d\left(1 - 0.59\frac{\rho f_y}{f_c'}\right)$$

# Balance Condition

From similar triangles at balance condition:

$$\frac{c}{d} = \frac{0.003}{0.003 + (f_y/E_s)} = \frac{0.003}{0.003 + (f_y/29 \times 10^6)}$$

$$c = \frac{87{,}000}{87{,}000 + f_y} d$$




Strain diagram for balanced condition.

Use equation for a. Substitute into c = a / $\beta_1$

$$a = \frac{\rho f_y d}{0.85 f_c'}$$

$$\rho = \frac{A_s}{bd}$$

$$c = \frac{a}{\beta_1} = \frac{\rho f_y d}{0.85 \beta_1 f_c'}$$

Equate expressions for c:

$$\frac{\rho f_y d}{0.85 \beta_1 f_c'} = \frac{87{,}000}{87{,}000 + f_y} d$$

$$\rho_b = \left(\frac{0.85 \beta_1 f_c'}{f_y}\right)\left(\frac{87{,}000}{87{,}000 + f_y}\right)$$

**Table A.8** Balanced Ratio of Reinforcement $\rho_b$ for Rectangular Sections with Tension Reinforcement Only

| $f_y$ \ $f_c'$ | | 2,500 psi (17.2 MPa) $\beta_1 = 0.85$ | 3,000 psi (20.7 MPa) $\beta_1 = 0.85$ | 4,000 psi (27.6 MPa) $\beta_1 = 0.85$ | 5,000 psi (34.5 MPa) $\beta_1 = 0.80$ | 6,000 psi (41.4 MPa) $\beta_1 = 0.75$ |
|---|---|---|---|---|---|---|
| Grade 40 | $\rho_b$ | 0.0309 | 0.0371 | 0.0495 | 0.0582 | 0.0655 |
| 40,000 psi | $0.75\rho_b$ | 0.0232 | 0.0278 | 0.0371 | 0.0437 | 0.0492 |
| (275.8 MPa) | $0.50\rho_b$ | 0.0155 | 0.0186 | 0.0247 | 0.0291 | 0.0328 |
| Grade 50 | $\rho_b$ | 0.0229 | 0.0275 | 0.0367 | 0.0432 | 0.0486 |
| 50,000 psi | $0.75\rho_b$ | 0.0172 | 0.0206 | 0.0275 | 0.0324 | 0.0365 |
| (344.8 MPa) | $0.50\rho_b$ | 0.0115 | 0.0138 | 0.0184 | 0.0216 | 0.0243 |
| Grade 60 | $\rho_b$ | 0.0178 | 0.0214 | 0.0285 | 0.0335 | 0.0377 |
| 60,000 psi | $0.75\rho_b$ | 0.0134 | 0.0161 | 0.0214 | 0.0252 | 0.0283 |
| (413.7 MPa) | $0.50\rho_b$ | 0.0089 | 0.0107 | 0.0143 | 0.0168 | 0.0189 |
| Grade 75 | $\rho_b$ | 0.0129 | 0.0155 | 0.0207 | 0.0243 | 0.0274 |
| 75,000 psi | $0.75\rho_b$ | 0.0097 | 0.0116 | 0.0155 | 0.0182 | 0.0205 |
| (517.1 MPa) | $0.50\rho_b$ | 0.0065 | 0.0078 | 0.0104 | 0.0122 | 0.0137 |



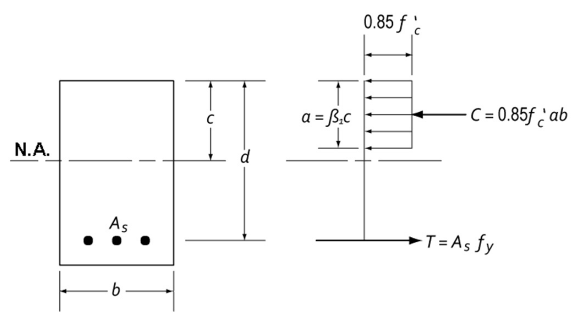
# Rectangular Beam Analysis

Data:

- Section dimensions – b, h, (span)
- Steel area - As
- Material properties – f'c, fy

Required:

- Nominal Strength (of beam) Moment - Mn
- Required (by load) Design Moment – Mu
- Load capacity




$As_{min}$:
greater of (a) and (b)

(a) $\frac{3\sqrt{f_c'}}{f_y} b_w d$

(b) $\frac{200}{f_y} b_w d$

$$c = \frac{a}{\beta_1}$$

$$\varepsilon_t = \frac{d-c}{c} 0.003 \geq 0.005$$

$$a = \frac{A_s f_y}{0.85 f_c' b}$$

$$M_n = A_s f_y \left(d - \frac{a}{2}\right)$$

$$\varphi M_n \geq M_u$$

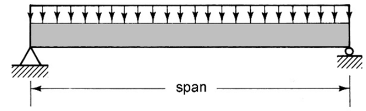
$$M_u = \frac{(1.2 w_{DL} + 1.6 w_{LL}) l^2}{8}$$

$$1.6 w_{LL} = \frac{M_u 8}{l^2} - 1.2 w_{DL}$$

1. Calculate d
2. Check As min
3. Calculate a
4. Determine c
5. Check that $\varepsilon_t \geq 0.005$ (tension controlled)
6. Find nominal moment, Mn
7. Calculate required moment, $\phi$ Mn ≥ Mu

   (if $\varepsilon_t \geq 0.005$ then $\phi = 0.9$)
8. Determine max. loading (or span)

# Rectangular Beam Analysis

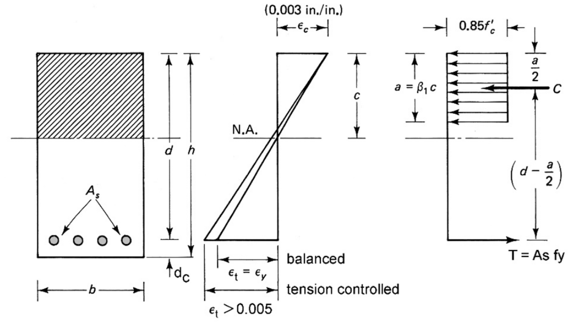



**Data:**

- Dimensions – 12”x23”
- Steel – 4 x # 6  fy = 60ksi
- Concrete f’c = 6000 psi
- Stirrup # 3, Cover 1.5”

**Required:**

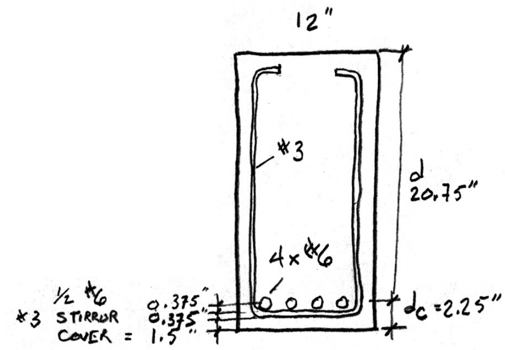
- Required Moment – $\phi$Mn = Mu (capacity)




1. Calculate d

$$d_c = \text{COVER} + \#3 + \tfrac{1}{2}(\#6)$$
$$= 1.5 + 0.375 + \frac{0.75}{2} = 2.25''$$

$$d = h - d_c = 23'' - 2.25 = 20.75''$$






# Rectangular Beam Analysis cont.

Data:
dimensions – 12”x23”
Steel – 4 x # 6 – As = 1.76 in$^2$
f’c = 6000 psi fy = 60ksi

**Table A.2** Designations, Areas, Perimeters, and Weights of Standard Bars

| | Customary Units | | | SI Units | | |
|---|---|---|---|---|---|---|
| Bar No. | Diameter (in.) | Cross-sectional Area (in.$^2$) | Unit Weight (lb/ft) | Diameter (mm) | Cross-sectional Area (mm$^2$) | Unit Weight (kg/m) |
| 3 | 0.375 | 0.11 | 0.376 | 9.52 | 71 | 0.560 |
| 4 | 0.500 | 0.20 | 0.668 | 12.70 | 129 | 0.994 |
| 5 | 0.625 | 0.31 | 1.043 | 15.88 | 200 | 1.552 |
| 6 | 0.750 | 0.44 | 1.502 | 19.05 | 284 | 2.235 |
| 7 | 0.875 | 0.60 | 2.044 | 22.22 | 387 | 3.042 |
| 8 | 1.000 | 0.79 | 2.670 | 25.40 | 510 | 3.973 |
| 9 | 1.128 | 1.00 | 3.400 | 28.65 | 645 | 5.060 |
| 10 | 1.270 | 1.27 | 4.303 | 32.26 | 819 | 6.404 |
| 11 | 1.410 | 1.56 | 5.313 | 35.81 | 1006 | 7.907 |
| 14 | 1.693 | 2.25 | 7.650 | 43.00 | 1452 | 11.384 |
| 18 | 2.257 | 4.00 | 13.600 | 57.33 | 2581 | 20.238 |

2. Check $As_{min}$

As min

① $\frac{3\sqrt{f'c}}{f_y} bd = \frac{3\sqrt{6000}}{60000}(12 \times 20.75) = 0.964 \text{ in}^2$ ← CONTROLS

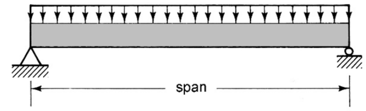
② $\frac{200\, bd}{f_y} = \frac{200(12)(20.75)}{60000} = 0.83 \text{ in}^2$

$\therefore A_{s\,min} = 0.964 \text{ in}^2$

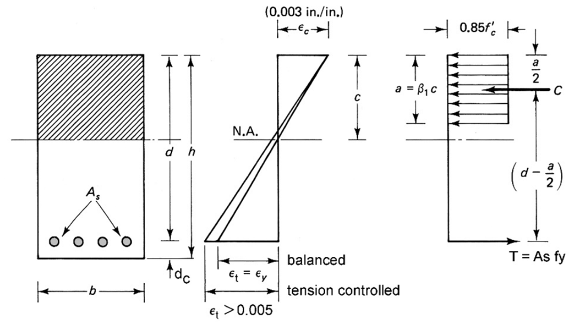
$A_s = A_b \text{ (N.B.)} = 0.44(4) = 1.76 \text{ in}^2 \checkmark > 0.964 \text{ in}^2$

# Rectangular Beam Analysis cont.

Data:
dimensions – 12”x23”
Steel – 4 x # 6 – As = 1.76 in$^2$
f’c = 6000 psi  fy = 60ksi




| f'$_c$ | $\beta_1$ |
|---|---|
| 0 | 0.85 |
| 1000 | 0.85 |
| 2000 | 0.85 |
| 3000 | 0.85 |
| 4000 | 0.85 |
| 5000 | 0.8 |
| 6000 | 0.75 |
| 7000 | 0.7 |
| 8000 | 0.65 |
| 9000 | 0.65 |
| 10000 | 0.65 |




Beta1:

$$0.85 \geq 0.85 - 0.05\left(\frac{f_c' - 4000}{1000}\right) \geq 0.65$$

3. Find a

$$a = \frac{A_s f_y}{0.85 f'_c b} = \frac{(1.76)(60)}{.85(6)(12)} = 1.725''$$

4. Find c

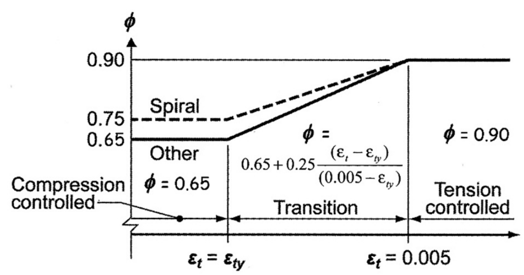
$$\beta_1 = 0.85 - 0.05\,\frac{f'_c - 4000}{1000} = 0.85 - 0.1 = 0.75$$

$$c = a/\beta_1 = \frac{1.725}{0.75} = 2.300''$$



# Rectangular Beam Analysis cont.

$$\varepsilon_t = \frac{d-c}{c}0.003 \geq 0.005$$




5. Check that $\varepsilon_t \geq 0.005$
(for tension controlled section)
$\phi$ = 0.9

$$\epsilon_t = \frac{d-c}{c}0.003 = \frac{20.75-2.3}{2.3}0.003$$
$$= 0.02406 > 0.004 \;\therefore\; \text{ok} ✓$$
$$= 0.02406 > 0.005 \;\therefore\; \text{tension controlled}$$

6. Find nominal moment, Mn

$$T = A_s f_y = 1.76^{in^2}(60^{ksi}) = 105.6\ K$$

$$M_n = T\left(d - \frac{a}{2}\right) = 105.6\left(20.75 - \frac{1.725''}{2}\right)$$
$$M_n = 2100\ K\text{-}''$$

7. Calculate required moment
$\phi$Mn ≥ Mu

$$\phi M_n = 0.9(2100) = 1890\ K\text{-}''$$

$$M_U = \phi M_n = 1890/12 = \underline{157.5}\ K\text{-}'$$