Architecture 324
Structures II

Structural Continuity

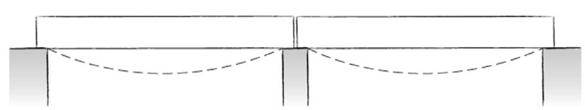
- Continuity in Beams
- Deflection Method
- Slope Method
- Three-Moment Theorem

Millennium Bridge, London
Foster and Partners + Arup

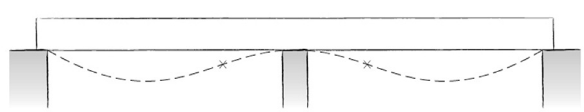
Photo by Ryan Donaghy

Continuous Beams

- Continuous over one or more supports
  - Most common in monolithic concrete
  - Steel: continuous or with moment connections
  - Wood: as continuous beams, e.g. long Glulam spans
- Statically indeterminate
  - Cannot be solved by the three equations of statics alone
  - Internal forces (shear & moment) as well as reactions are affected by movement or settlement of the supports

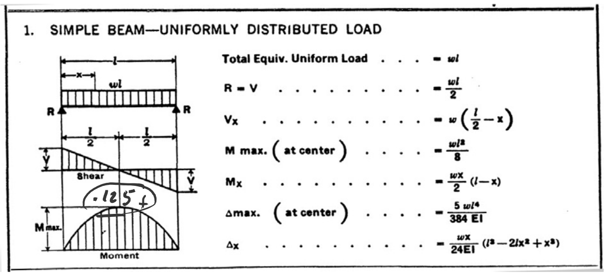


two spans - simply supported



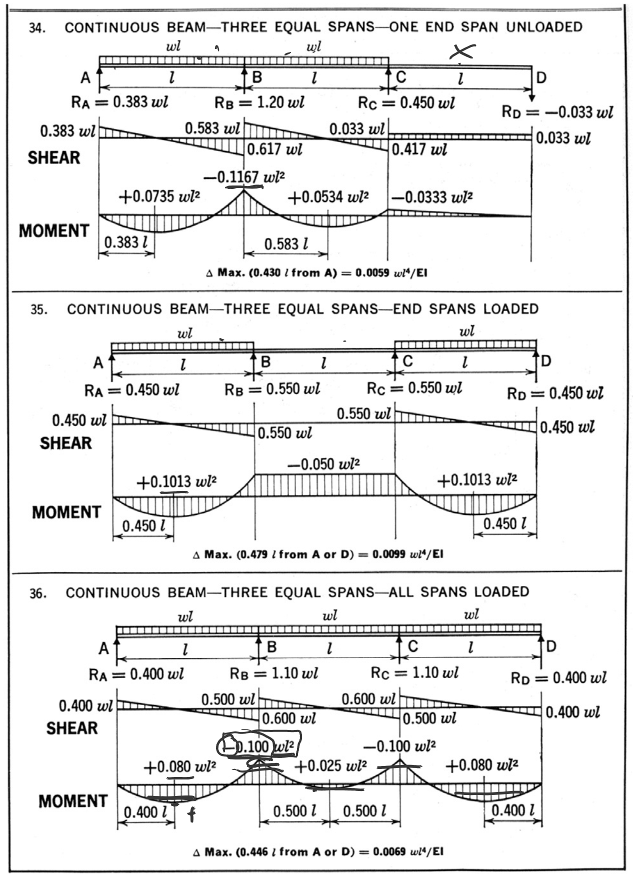
two spans - continuous

Simple vs. Continuous Beams



- Simple Beam
  - End moments = 0
  - Mmax at C.L = $wL^2/8 = 0.125wL^2$
- Continuous Beam
  - Exterior end moments = 0
  - Interior support moments are usually negative
  - Mid-span moments are usually positive
  - End + Mid = $0.125wL^2$

Note: moments shown reversed

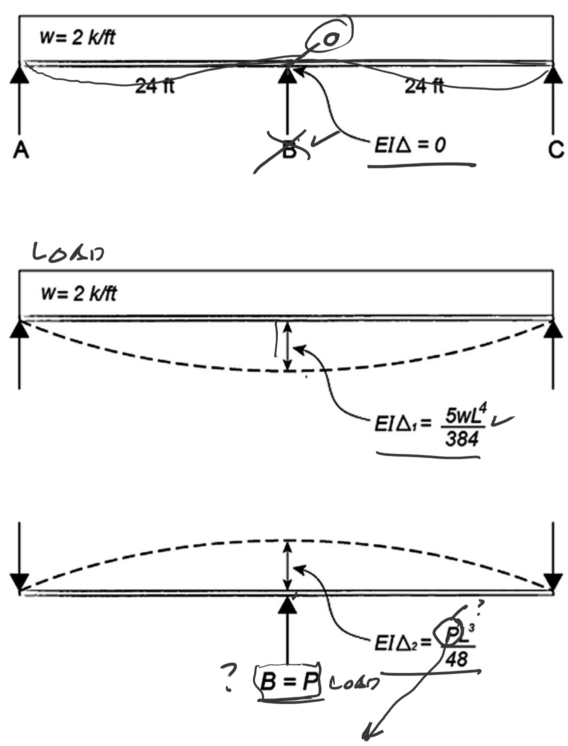


Deflection Method

- Two continuous, symmetric spans
- Symmetric Load

Procedure:

1. Remove the center support, and calculate the center deflection for each load case as a simple span.
2. Remove the applied loads and replace the center support. Set the deflection equation for this case (center point load) equal to the deflection from step 1.
3. Solve the resulting equation for the center reaction force. (upward point load)
4. Calculate the remaining two end reactions.
5. Draw shear and moment diagrams as usual.

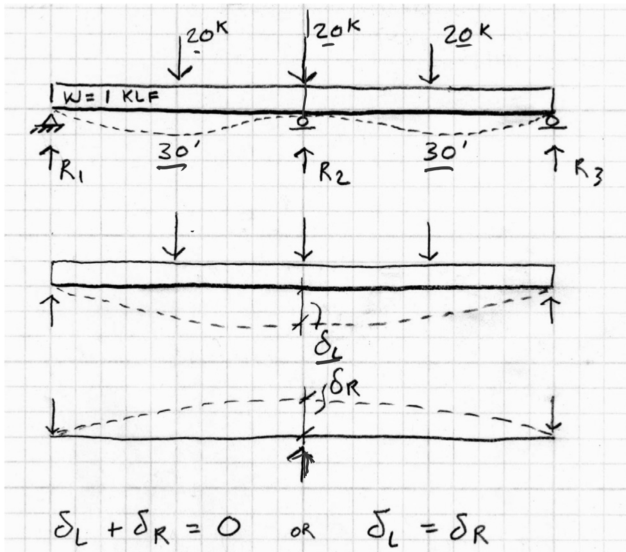


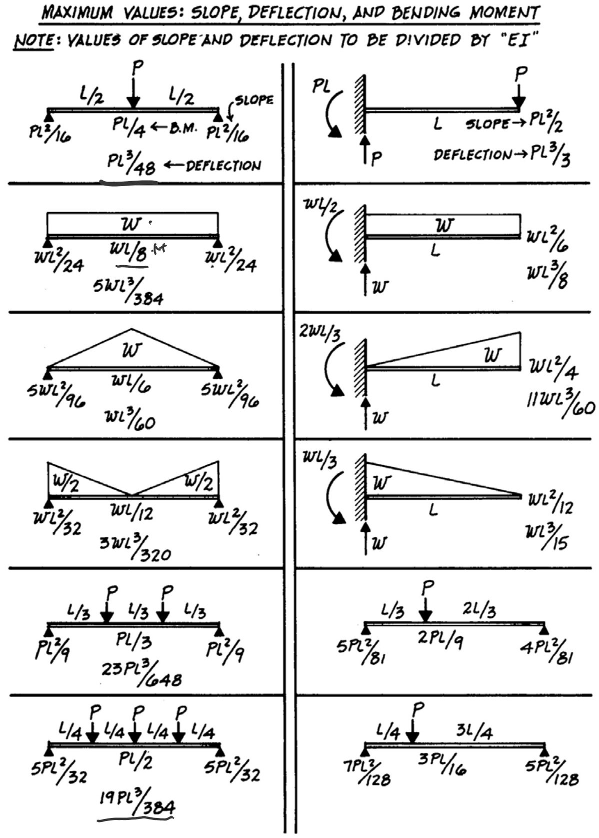
$$EI\Delta_1 + EI\Delta_2 = 0$$

Deflection Method - Example:

Given: Two symmetric spans with symmetric loading as shown.

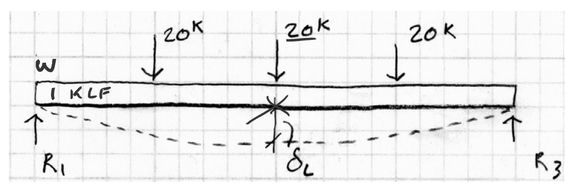
Find: All three reactions

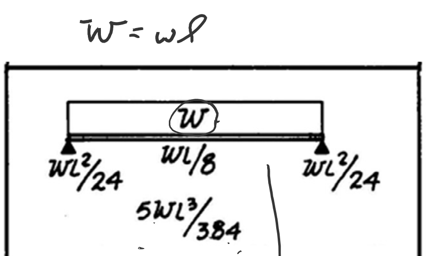


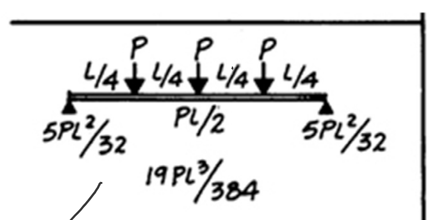


Deflection Method

1. Remove the center support, and calculate the center deflection for each load case as a simple span.





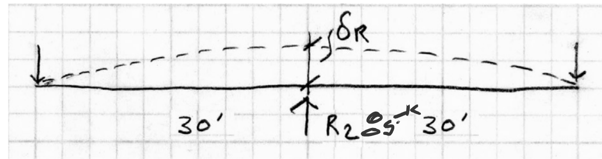


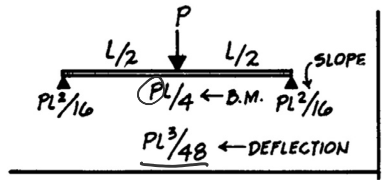
$$\delta_L = \frac{5wl^4}{384\,EI} + \frac{19\,Pl^3}{384\,EI} = \frac{5(1)(60)^4 + 19(20)(60)^3}{384\,EI}$$

$$\delta_L = 382500/EI$$

Deflection Method – Example

2. Remove the applied loads and replace the center support. Set the deflection equation for this case (center point load) equal to the deflection from step 1.





3. Solve the resulting equation for the center reaction force. (upward point load)

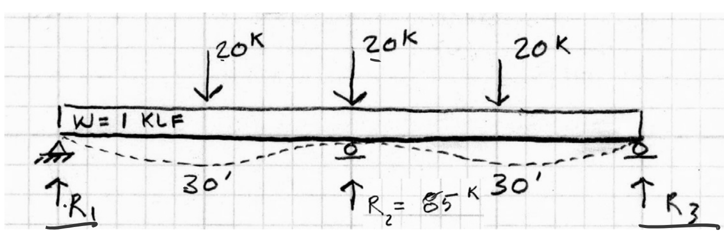
$$\delta_L = \delta_R$$

$$382500/\cancel{EI} = \frac{R_2 \ell^3}{48\,\cancel{EI}}$$

$$R_2 = 85^K$$

Deflection Method – Example

4. Calculate the remaining two end reactions.



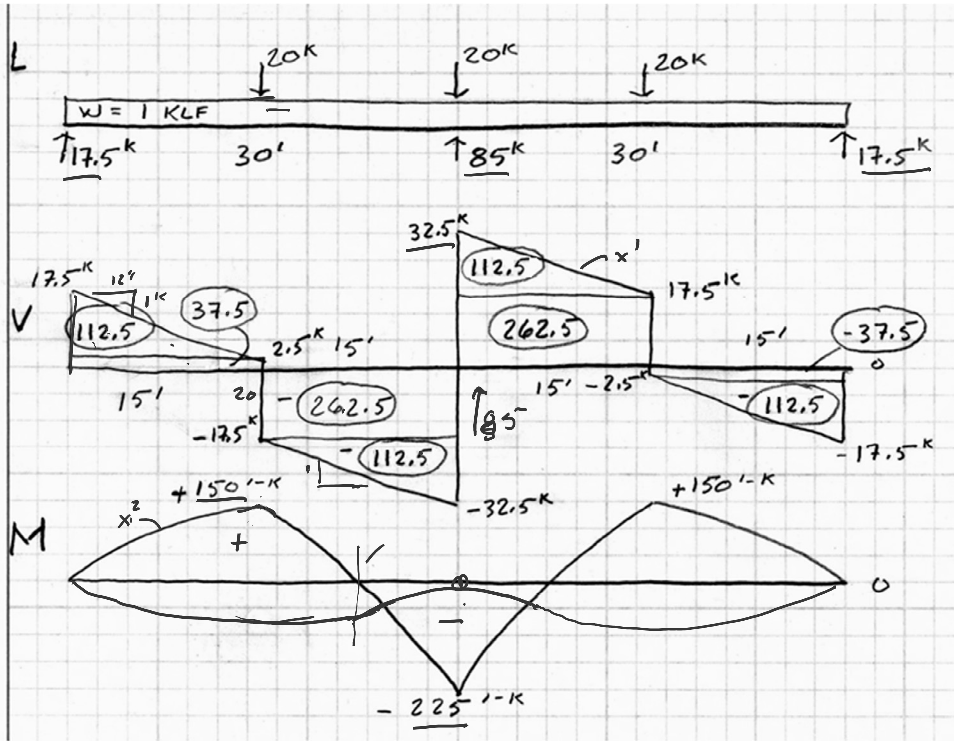
3(20) 1(60)

$$\Sigma F_v = 0 = R_1 + R_3 + 85 - 60 - 60 = 0$$

$$R_1 = R_3 = 17.5^K$$

Deflection Method - Example cont.:

5. Draw shear and moment diagrams as usual.

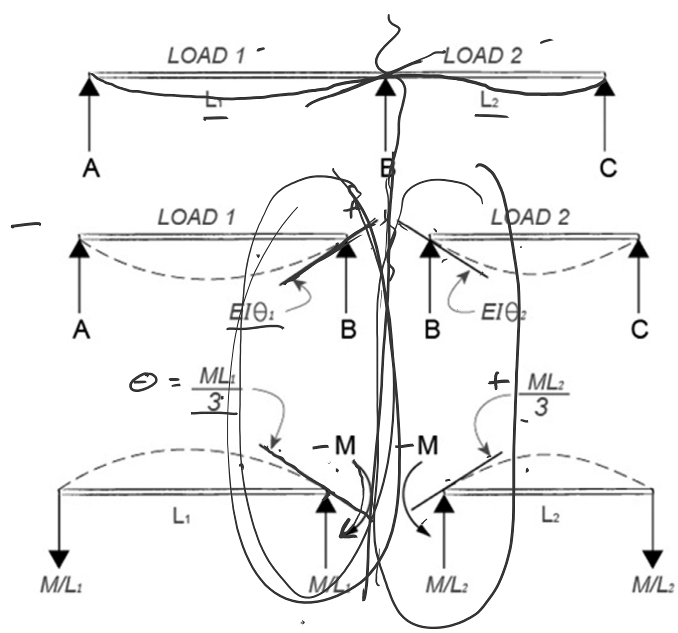


Slope Method

- Two continuous spans
- Non-symmetric loads and spans

Procedure:

1. Break the beam into two halves at the interior support, and calculate the interior slopes of the two simple spans.
2. Use the Slope Equation to solve for the negative interior moment.
3. Find the reactions of each of the simple spans plus the M/L reactions caused by the interior moment.
4. Add all the reactions by superposition.
5. Draw the shear and moment diagrams as usual.



SLOPE = SLOPE

$$+EI\theta_1 - \frac{ML_1}{3} = -EI\theta_2 + \frac{ML_2}{3}$$

$$\frac{ML_1}{3} + \frac{ML_2}{3} = (EI\theta_1 + EI\theta_2)$$

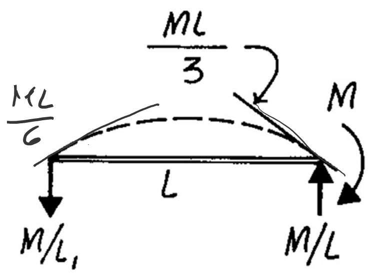
$$M = \frac{3}{L_1+L_2}(EI\theta_1 + EI\theta_2)$$

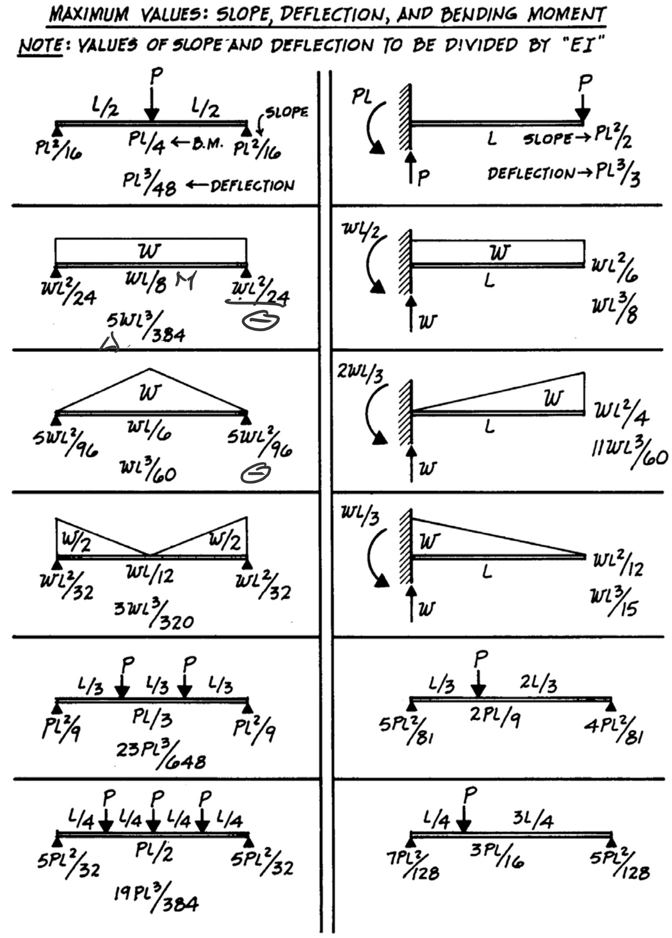
$$M = \frac{3}{L_1 + L_2}\left[EI\Theta_1 + EI\Theta_2\right]$$

Slope Method

Slope equations:

$$M = \frac{3}{L_1 + L_2}\left[EI\Theta_1 + EI\Theta_2\right]$$

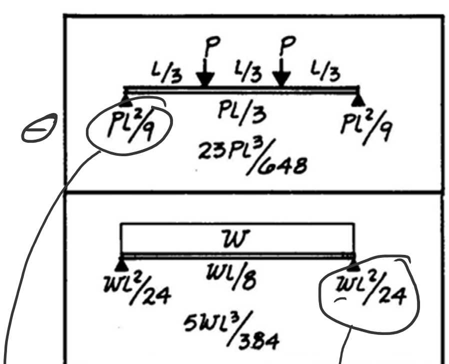




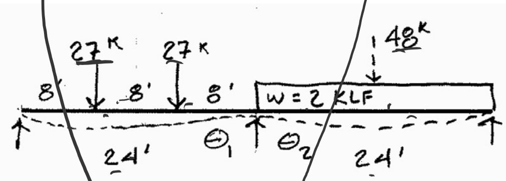
Slope Method - Example

Given: Two non-symmetric spans with loading as shown.

Find: All three reactions



1. Break the beam into two halves at the interior support, and calculate the interior slopes of the two simple spans.



2. Use the Slope Equation to solve for the negative interior moment.

$$M = \frac{3}{L_1 + L_2}\left[EI\Theta_1 + EI\Theta_2\right]$$

$$EI\Theta_1 = \frac{PL^2}{9} = \frac{27(24)^2}{9} = 1728$$

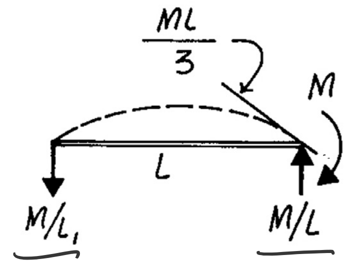
$$EI\Theta_2 = \frac{wL^2}{24} = \frac{48(24)^2}{24} = 1152$$

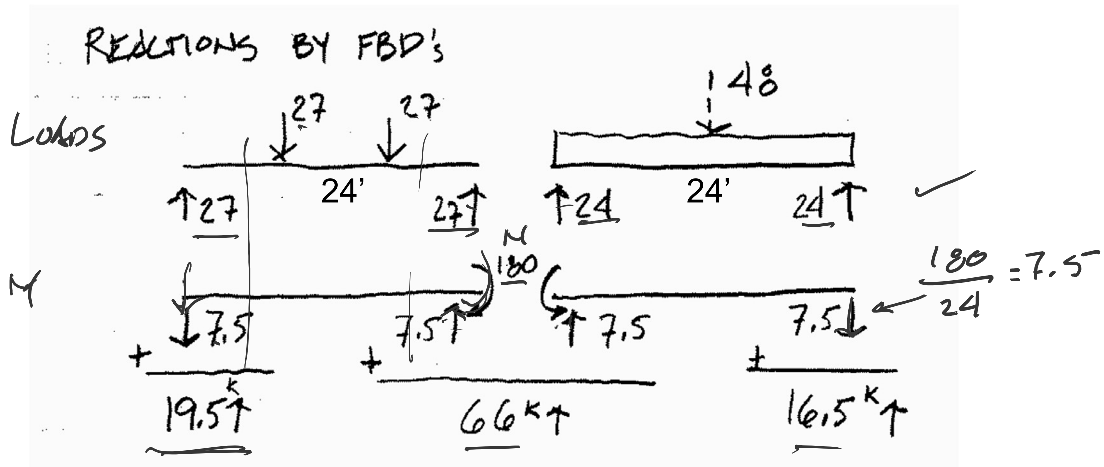
$$M = \frac{3}{L_1+L_2}\left[EI\Theta_1 + EI\Theta_2\right] = \frac{3}{48}\left[2880\right]$$

$$M = 180^{k\prime}$$

Example of Slope Method cont.:

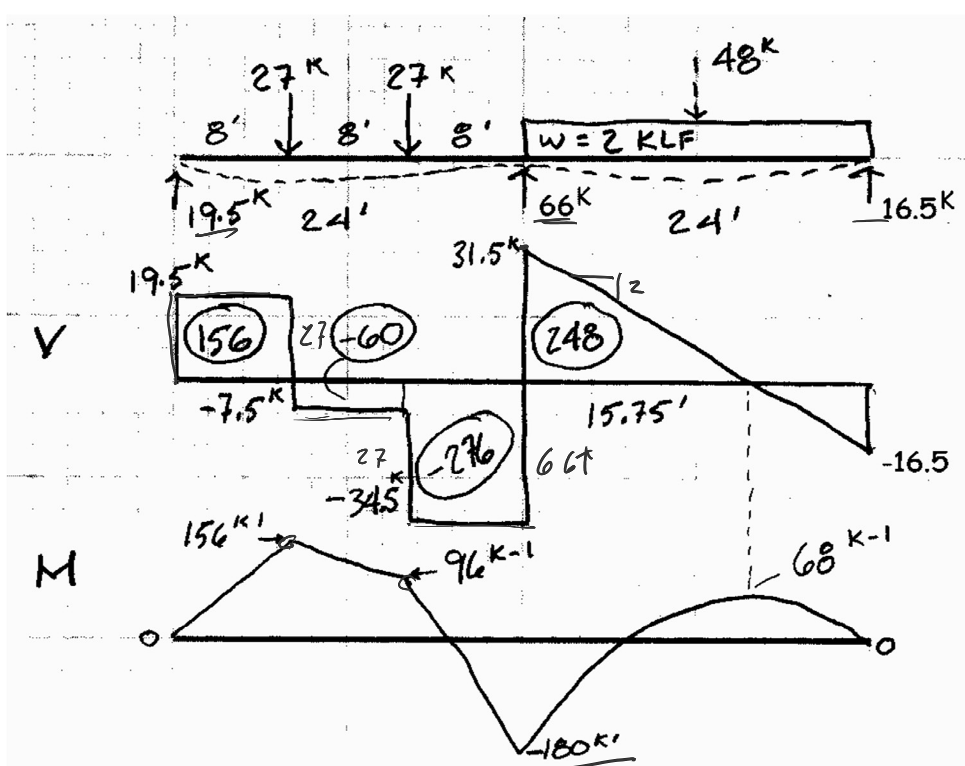
3. Find the reactions of each of the simple spans plus the M/L reactions caused by the interior moment.
4. Add all the reactions by superposition.





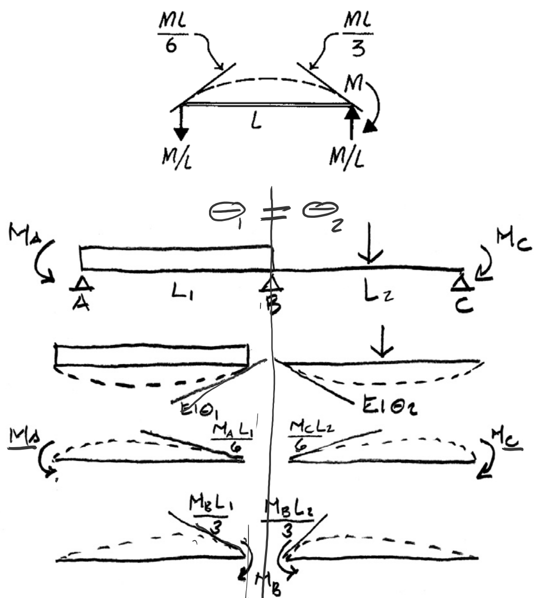
Example of Slope Method cont.:

5. Draw the shear and moment diagrams as usual.



3-Moment Theorem

- Any number of continuous spans
- Non-Symmetric Load and Spans

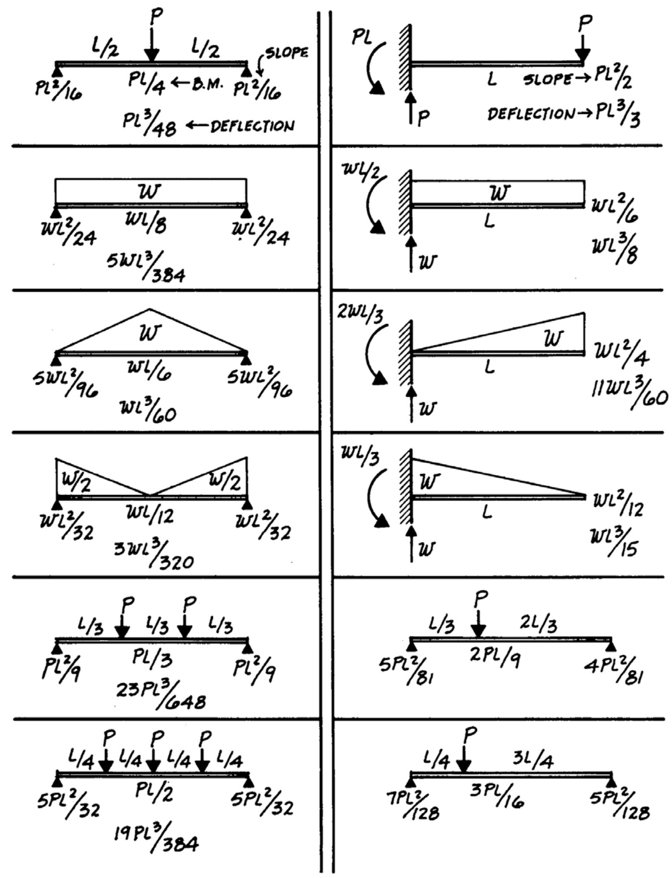


$$+EI\theta_1 - \frac{M_A L_1}{6} - \frac{M_B L_1}{3} = -EI\theta_2 + \frac{M_C L_2}{6} + \frac{M_B L_2}{3}$$

$$M_A L_1 + 2M_B(L_1 + L_2) + M_C L_2 = 6[EI\Theta_1 + EI\Theta_2]$$

MAXIMUM VALUES: SLOPE, DEFLECTION, AND BENDING MOMENT

NOTE: VALUES OF SLOPE AND DEFLECTION TO BE DIVIDED BY "EI"

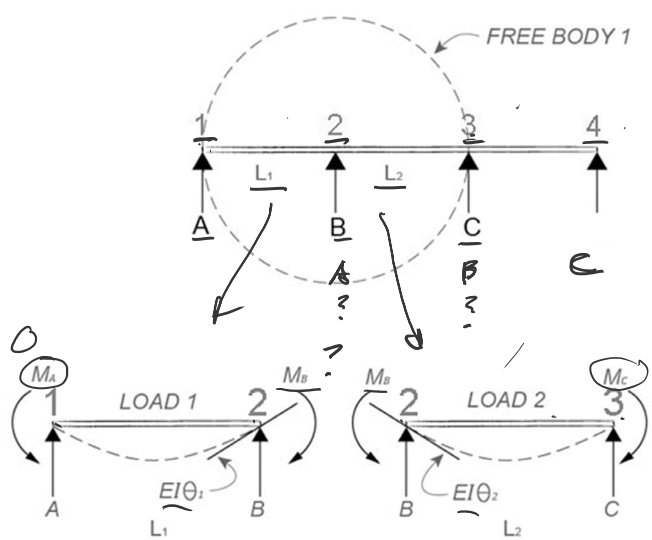


Three-Moment Theorem

- Any number of spans
- Symmetric or non-symmetric

Procedure:

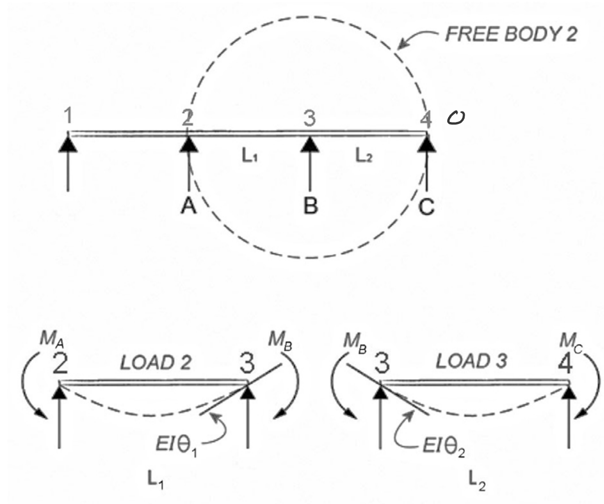
1. Draw a free body diagram of the first two spans.
2. Label the spans L1 and L2 and the supports (or free end) A, B and C as show.
3. Use the Three-Moment equation to solve for each unknown moment, either as a value or as an equation.



$$\overset{0}{M_A} L_1 + 2\overset{?}{M_B}(L_1 + L_2) + \overset{?}{M_C} L_2 = 6[EI\Theta_1 + EI\Theta_2]$$

Three-Moment Theorem
Procedure (continued):

4. Move one span further and repeat the procedure.
5. In a 3 span beam, the mid-moment from step 3 above (B), can now be solved using the two equations from step 4 and 3 together, by writing 2 equations with 2 unknowns.
6. Repeat as needed, always moving one span to the right and writing a new set of moment equations.
7. Solve 2 simultaneous equations for 3 spans, or 3 equations for more than 3 spans, to get the interior moments.
8. Once all interior moments are known, solve for reactions using free body diagrams of individual spans.
9. Draw shear and moment diagrams as usual. This will also serve as a check for the moment values.



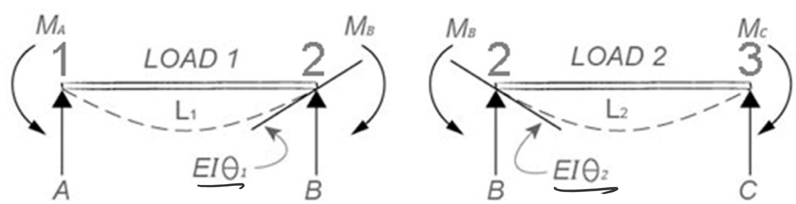
$$M_A L_1 + 2M_B\left(L_1 + L_2\right) + M_C L_2 = 6\left[EI\Theta_1 + EI\Theta_2\right]$$

2 3 0

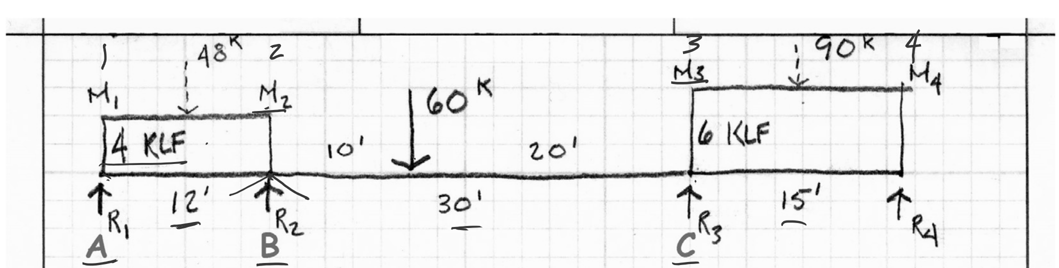
Three-Moment Theorem Example

Given: Three non-symmetric spans with loading as shown.

Find: All four reactions



1. Draw FBD
2. Label
3. Solve 3-moment equation



FIRST EQUATION (AT 1, 2 AND 3):

$M_A = 0$ $\quad L_1 = 12$ $\quad EI\Theta_1 = \frac{w\ell^2}{24} = \frac{48(12)^2}{24} = 288$

$M_B = M_2?$ $\quad L_2 = 30$

$M_C = M_3?$ $\quad L_1 + L_2 = 42$ $\quad EI\Theta_2 = \frac{5P\ell^2}{81} = \frac{5(60)(30)^2}{81} = 3333.3$

$$M_A L_1 + 2M_B(L_1 + L_2) + M_C L_2 = 6[EI\Theta_1 + EI\Theta_2]$$

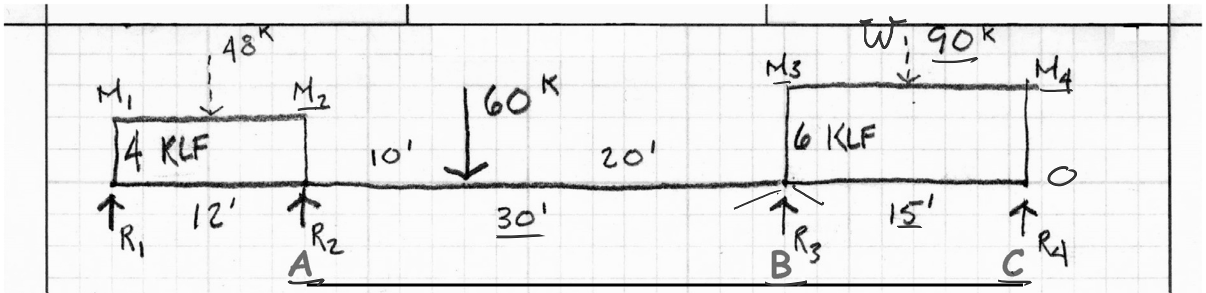
$$0(12) + 2(M_2)(42) + (M_3)(30) = 6[288 + 3333.3]$$

$$M_2 = 258.667 - 0.35714\, M_3$$

Three-Moment Theorem Example (cont.)

$$M_A L_1 + 2M_B(L_1 + L_2) + M_C L_2 = 6[EI\Theta_1 + EI\Theta_2]$$

4. Move one span further and repeat the procedure.



SECOND EQUATION (AT 2, 3 AND 4):

$M_A = M_2$? $L_1 = 30$ $EI\Theta_1 = \frac{4Pl^2}{81} = \frac{4(60)(30)^2}{81} = 2666.67$

$M_B = M_3$? $L_2 = 15$

$M_C = 0$ $L_1 + L_2 = 45$ $EI\Theta_2 = \frac{wl^2}{24} = \frac{90(15^2)}{24} = 843.75$

$$M_2(30) + 2M_3(45) + 0(15) = 6[2666.67 + 843.75]$$

$$M_2 = 702.084 - 3M_3$$

SIMULTANIOUS SOLUTION:

$$258.667 - .35714\,M_3 = 702.084 - 3M_3$$

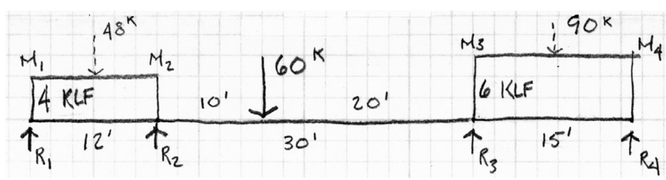
$$M_3 = 443.417/2.6428 = 167.779\ ^{K\text{-}'}$$

$$M_2 = 702.084 - 3(167.779) = 198.746\ ^{K\text{-}'}$$

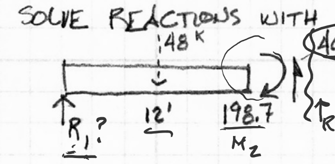
Three-Moment Theorem Example (cont.)

Sign convention

8. Once all interior moments are known, solve for reactions using free body diagrams of individual spans.



SOLVE REACTIONS WITH FBD's

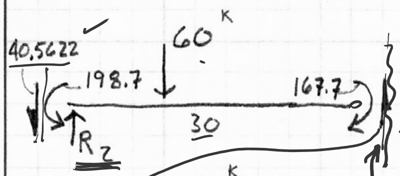


$\Sigma M@R_2 = R_1(12) - 48(6) + 198.7 = 0$

$R_1 = 7.4378^K$

$\Sigma F_V = 7.4378 - 48 + V = 0$

$V = 40.5622$

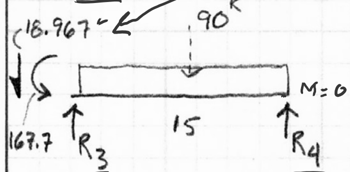


$\Sigma M@R_3 = R_2(30) - 40.5622(30) - 60(20) - 198.75 + 167.78 = 0$

$R_2 = 81.594^K$

$\Sigma F_V = -40.5622 + 81.594 - 60 + V = 0$

$V = 18.967$



$\Sigma M@R_4 = -167.78 - 18.967(15) + R_3(15) - 90(7.5) = 0$

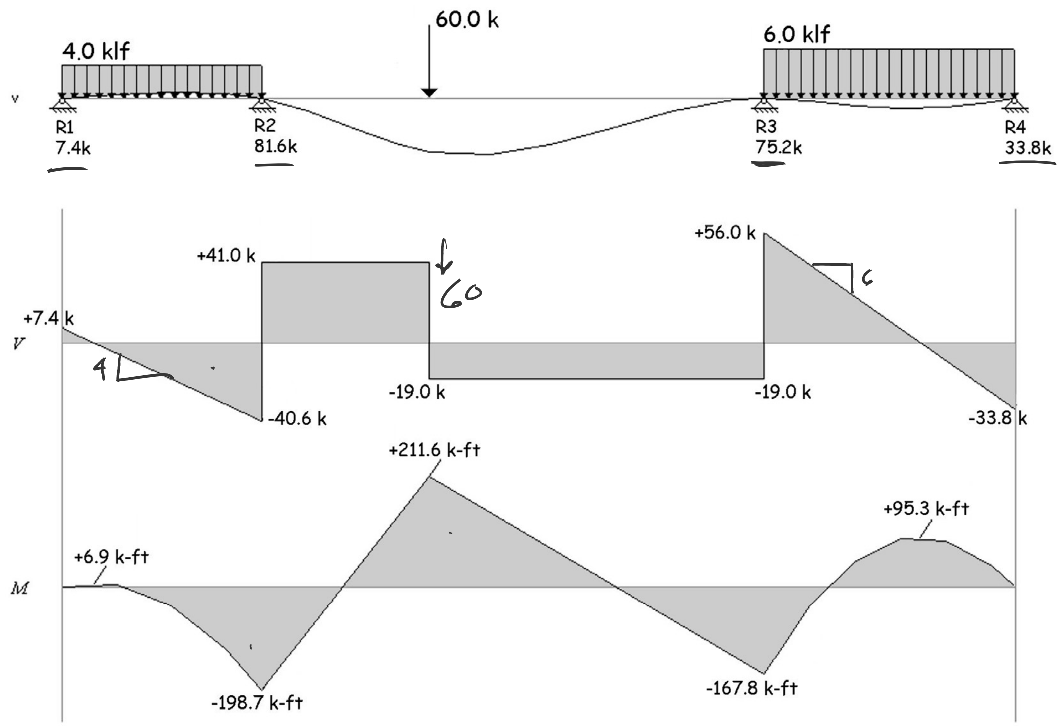
$R_3 = 75.1531^K$

$\Sigma M@R_3 = -167.78 + 90(7.5) - R_4(15) = 0$

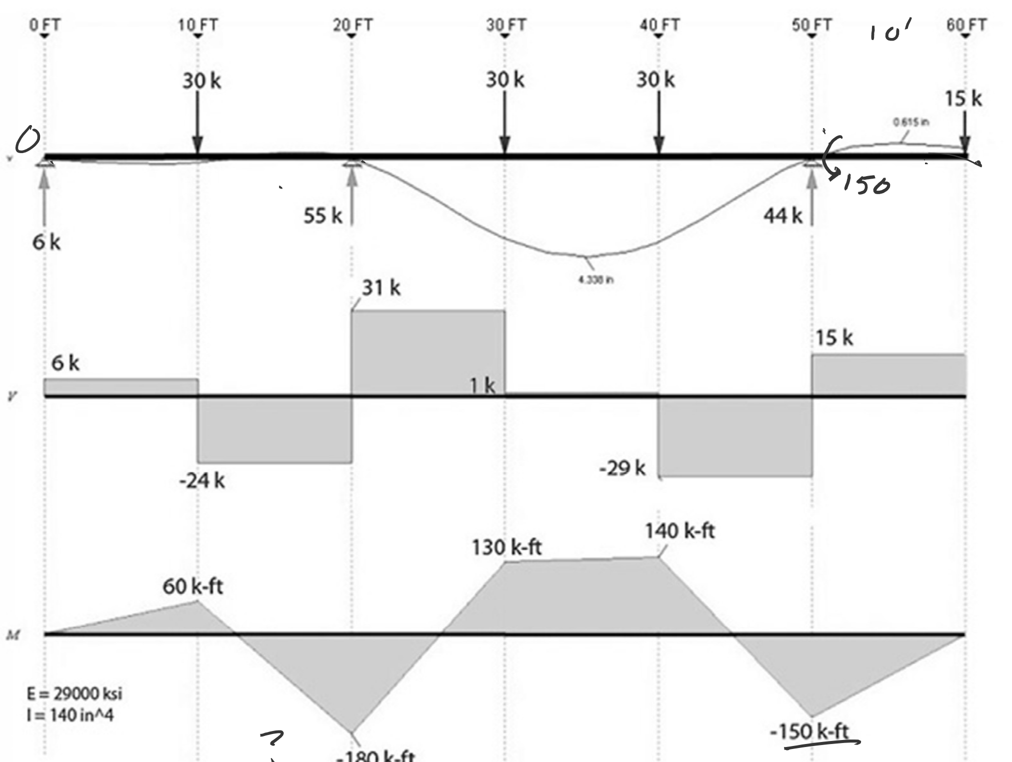
$R_4 = 33.8147^K$

Three-Moment Theorem Example (cont.)

9. Draw shear and moment diagrams as usual. This will also serve as a check for the moment values.



Three-Moment Theorem – 2 Spans

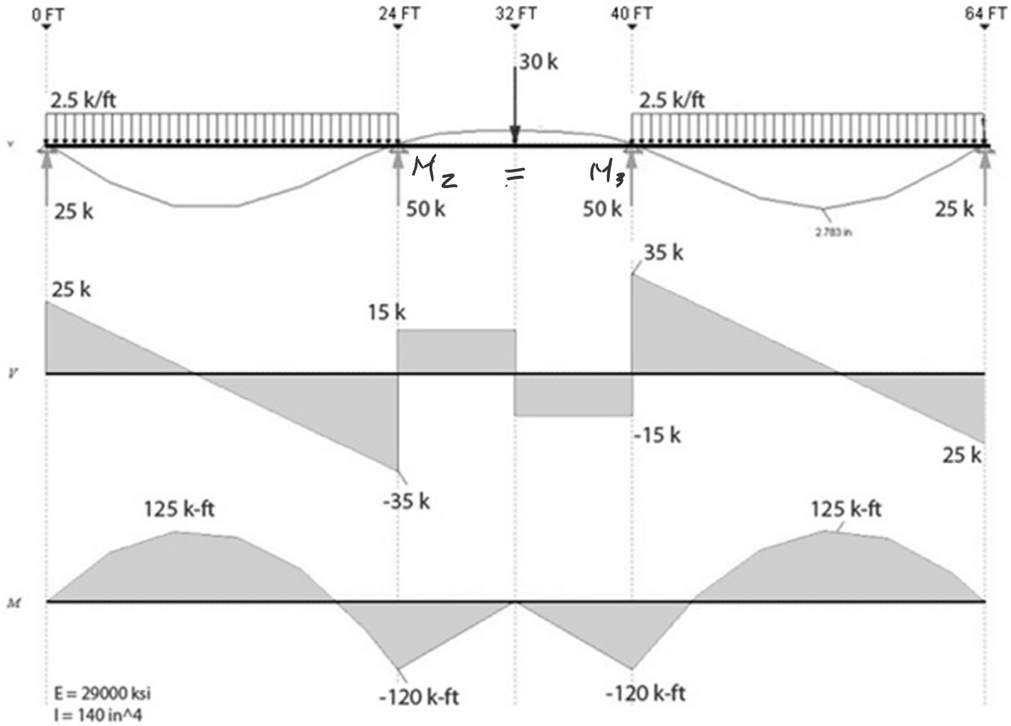


?

0 150

$$M_A L_1 + 2\boxed{M_B}\left(L_1 + L_2\right) + M_C L_2 = 6\left[EI\Theta_1 + EI\Theta_2\right]$$

Three-Moment Theorem – 3 Spans

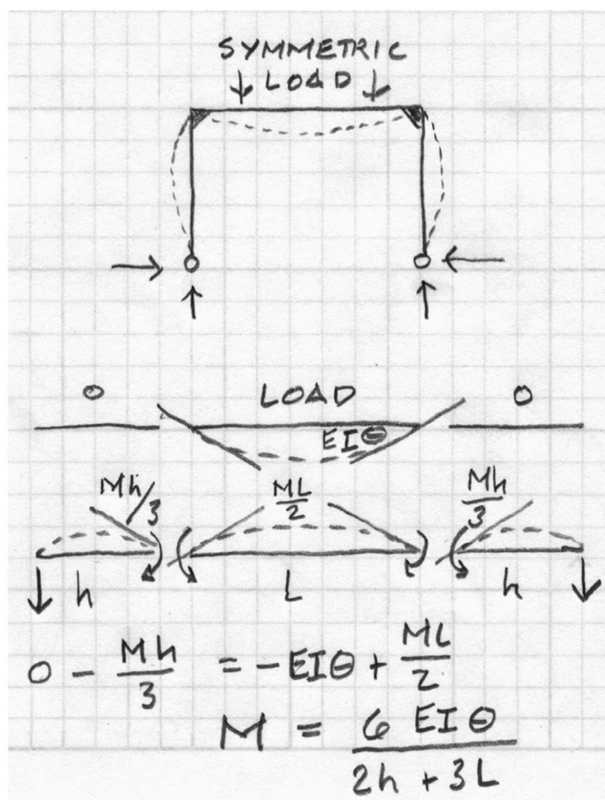


$$M_A L_1 + 2\underline{M_B}(L_1 + L_2) + \underline{M_C} L_2 = 6[EI\Theta_1 + EI\Theta_2]$$

2-Hinge Frame

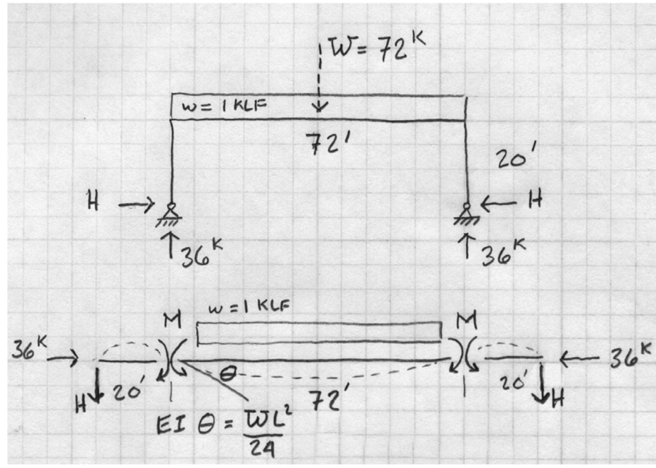
- Statically indeterminate
- Find negative moment at knee
- Symmetric case solution

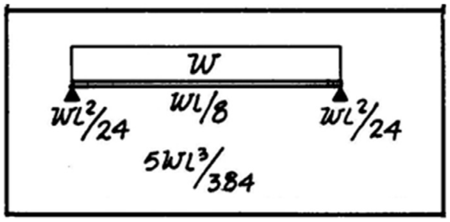
$$M = \frac{6 \ EI\Theta}{2h + 3L}$$



2-Hinge Frame example

- Symmetric case solution
- Vertical reactions by symmetry
- Find moment at knee
- With FBD of one leg find H



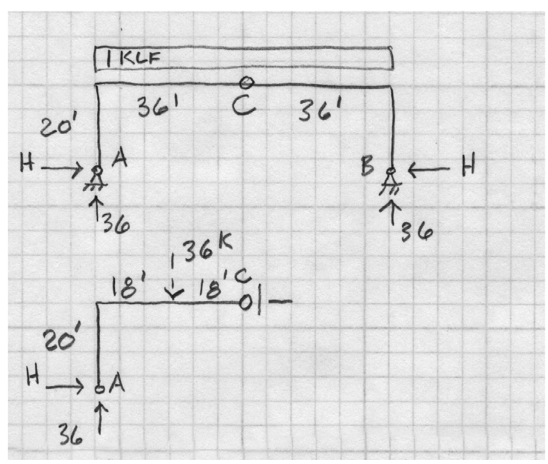


$$M = \frac{6\ EI\theta}{2h + 3L} = \frac{6\ \frac{72(72)^2}{24}}{2(20) + 3(72)} = \underline{364.5}^{K\text{-}I}$$

$$M = H(20), \quad H = M/20 = 364.5/20 = \underline{18.2}^{K}$$

3-Hinge Frame comparison

- Statically determinate
- Solve with statics
- FBD of half from hinge
- Solve for H
- Use FBD of leg to solve M



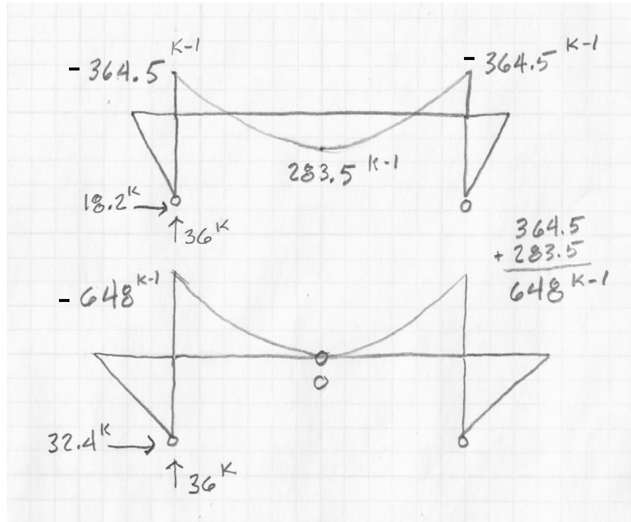
$$\Sigma M_C = 0 = -36(18) + 36(36) - H(20)$$

$$H = 32.4^{K}$$

$$M = H(20) = 32.4(20) = 648^{K\text{-}I}$$

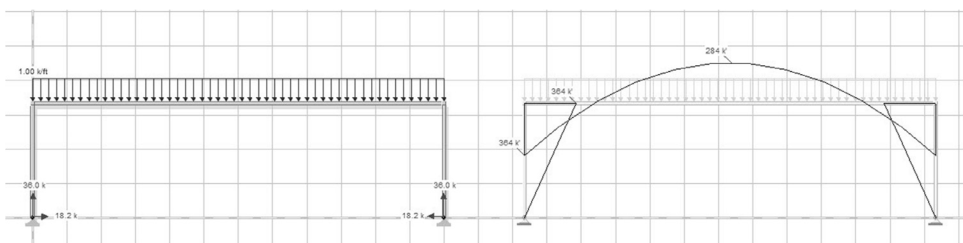
Comparison of moments

- 2-hinge frame
- 3-hinge frame



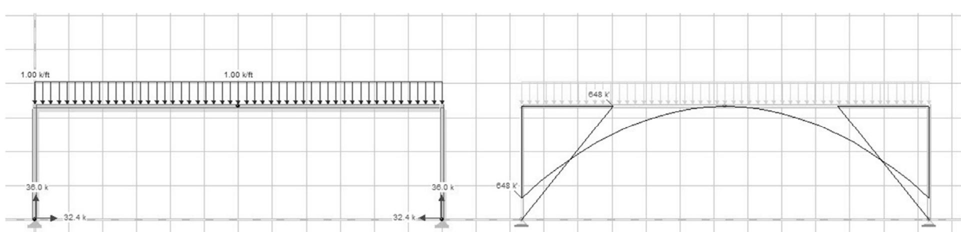
The effect of shape and hinges

Moment:
knee: -364 ft-lbs
center: +284 ft-lbs
horz. react. = 18.2 k



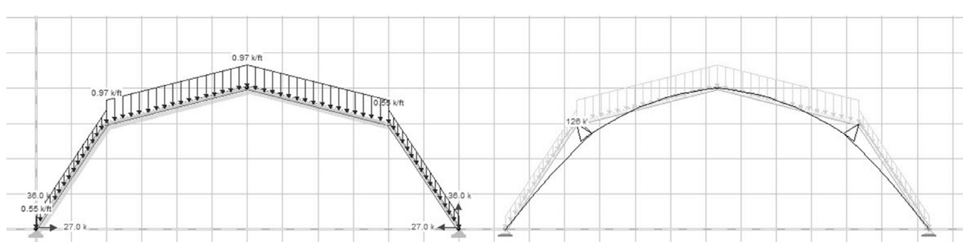
Continuous Beam

Moment:
knee: -648 ft-lbs
center: 0 ft-lbs
horz. react. = 32.4 k



Beam with center hinge

Moment:
knee: -126 ft-lbs
center: 0 ft-lbs
horz. react. = 27.0 k



3 Hinged Arch