

# 3-Moment Theorem

- Continuity in Beams
- Three-Moment Theorem
- 2-Hinged Frames
- 3-Hinged Frames
- 3-Hinged Arches

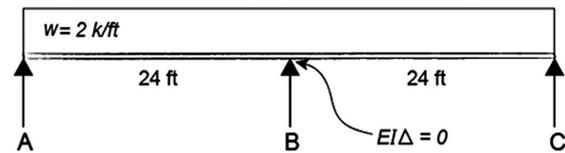


Salginatobel Bridge  
Switzerland, 1930  
300 ft span 42 ft rise  
Robert Maillart

# Continuous Beam Analysis

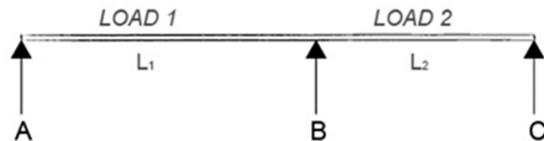
## Deflection Method

- Two continuous, symmetric spans
- Symmetric Load



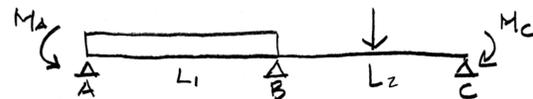
## Slope Method

- Two continuous spans
- Non-symmetric loads and spans



## 3-Moment Theorem

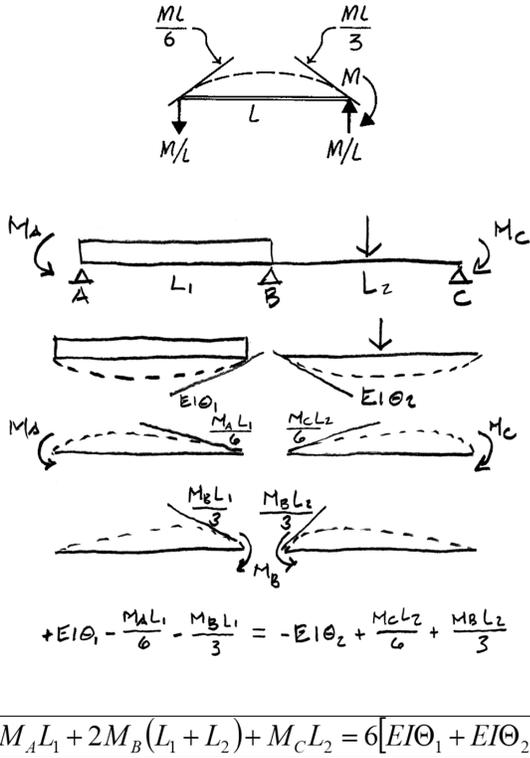
- Any number of continuous spans
- Non-Symmetric Load and Spans



$$M_A L_1 + 2M_B(L_1 + L_2) + M_C L_2 = 6[EI\theta_1 + EI\theta_2]$$

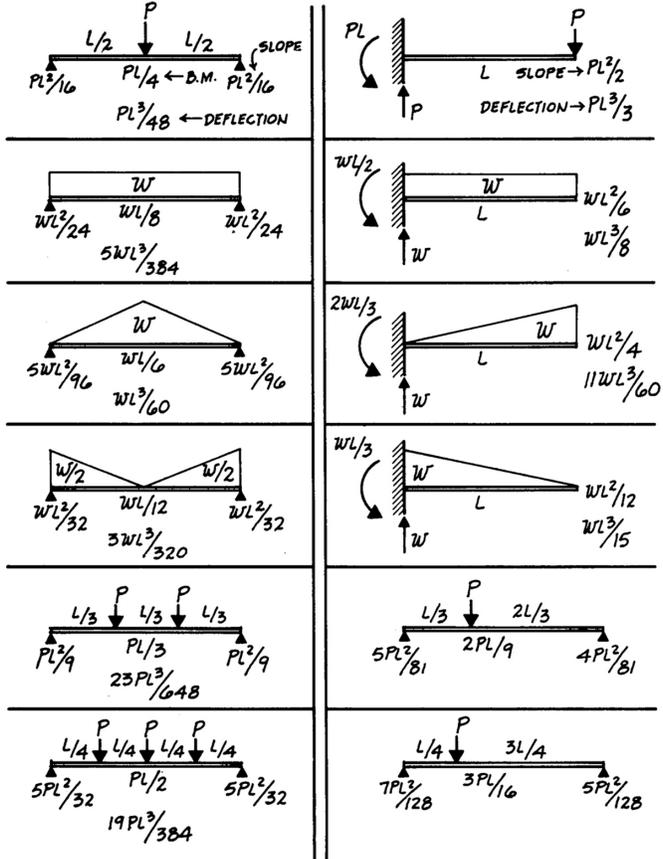
### 3-Moment Theorem

- Any number of continuous spans
- Non-Symmetric Load and Spans



### MAXIMUM VALUES: SLOPE, DEFLECTION, AND BENDING MOMENT

NOTE: VALUES OF SLOPE AND DEFLECTION TO BE DIVIDED BY "EI"

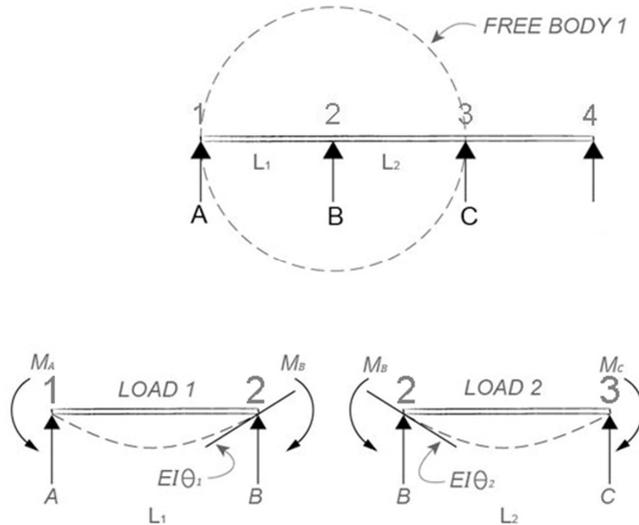


### Three-Moment Theorem

- Any number of spans
- Symmetric or non-symmetric

Procedure:

1. Draw a free body diagram of the first two spans.
2. Label the spans  $L_1$  and  $L_2$  and the supports (or free end) A, B and C as show.
3. Use the Three-Moment equation to solve for each unknown moment, either as a value or as an equation.

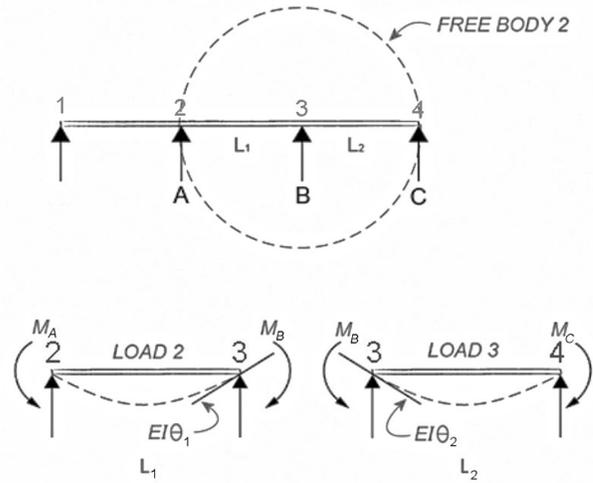


$$M_A L_1 + 2M_B (L_1 + L_2) + M_C L_2 = 6[EI\theta_1 + EI\theta_2]$$

# Three-Moment Theorem

Procedure (continued):

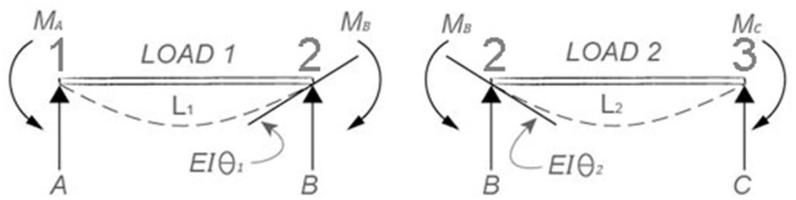
4. Move one span further and repeat the procedure.
5. In a 3 span beam, the mid-moment from step 3 above (B), can now be solved using the two equations from step 4 and 3 together, by writing 2 equations with 2 unknowns.
6. Repeat as needed, always moving one span to the right and writing a new set of moment equations.
7. Solve 2 simultaneous equations for 3 spans, or 3 equations for more than 3 spans, to get the interior moments.
8. Once all interior moments are known, solve for reactions using free body diagrams of individual spans.
9. Draw shear and moment diagrams as usual. This will also serve as a check for the moment values.



$$M_A L_1 + 2M_B (L_1 + L_2) + M_C L_2 = 6[EI\Theta_1 + EI\Theta_2]$$

## Three-Moment Theorem Example

Given: Three non-symmetric spans with loading as shown.



Find: All four reactions

1. Draw FBD
2. Label
3. Solve 3-moment equation

FIRST EQUATION (AT 1, 2 AND 3) :

$$M_A = 0 \quad L_1 = 12 \quad EI\Theta_1 = \frac{wL^2}{24} = \frac{48(12)^2}{24} = 288$$

$$M_B = M_2 \quad L_2 = 30 \quad EI\Theta_2 = \frac{5Pl^2}{81} = \frac{5(60)(30)^2}{81} = 3333.3$$

$$M_C = M_3 \quad L_1 + L_2 = 42$$

$$M_A L_1 + 2M_B (L_1 + L_2) + M_C L_2 = 6[EI\Theta_1 + EI\Theta_2]$$

$$0(12) + 2M_2(42) + M_3(30) = 6[288 + 3333.3]$$

$$M_2 = 258.667 - 0.35714 M_3$$

## Three-Moment Theorem Example (cont.)

$$M_A L_1 + 2M_B(L_1 + L_2) + M_C L_2 = 6[EI\theta_1 + EI\theta_2]$$

4. Move one span further and repeat the procedure.

SECOND EQUATION (AT 2, 3 AND 4) =

$$M_A = M_2 \quad L_1 = 30 \quad EI\theta_1 = \frac{4Pl^2}{81} = \frac{4(60)(30)^2}{81} = 2666.67$$

$$M_B = M_3 \quad L_2 = 15 \quad EI\theta_2 = \frac{wl^2}{24} = \frac{90(15^2)}{24} = 843.75$$

$$M_C = 0 \quad L_1 + L_2 = 45$$

$$M_2(30) + 2M_3(45) + 0(15) = 6[2666.67 + 843.75]$$

$$M_2 = 702.084 - 3M_3$$

SIMULTANEOUS SOLUTION:

$$258.667 - .35714 M_3 = 702.084 - 3M_3$$

$$M_3 = 443.417 / 2.6428 = 167.779 \text{ K}^{-1}$$

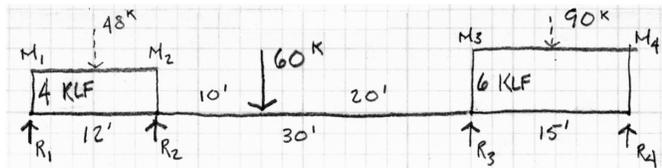
$$M_2 = 702.084 - 3(167.779) = 198.746 \text{ K}^{-1}$$

## Three-Moment Theorem Example (cont.)

Sign convention



8. Once all interior moments are known, solve for reactions using free body diagrams of individual spans.



SOLVE REACTIONS WITH FBD'S

$$\sum M_{R_2} = R_1(12) - 48(6) + 198.7 = 0$$

$$R_1 = 7.4378 \text{ K}$$

$$\sum F_v = 7.4378 - 48 + V = 0$$

$$V = 40.5622$$

$$\sum M_{R_3} = R_2(30) - 40.5622(30) - 60(20) - 198.75 + 167.78 = 0$$

$$R_2 = 81.594 \text{ K}$$

$$\sum F_v = -40.5622 + 81.594 - 60 + V = 0$$

$$V = 18.967$$

$$\sum M_{R_4} = -167.78 - 18.967(15) + R_3(15) - 90(7.5) = 0$$

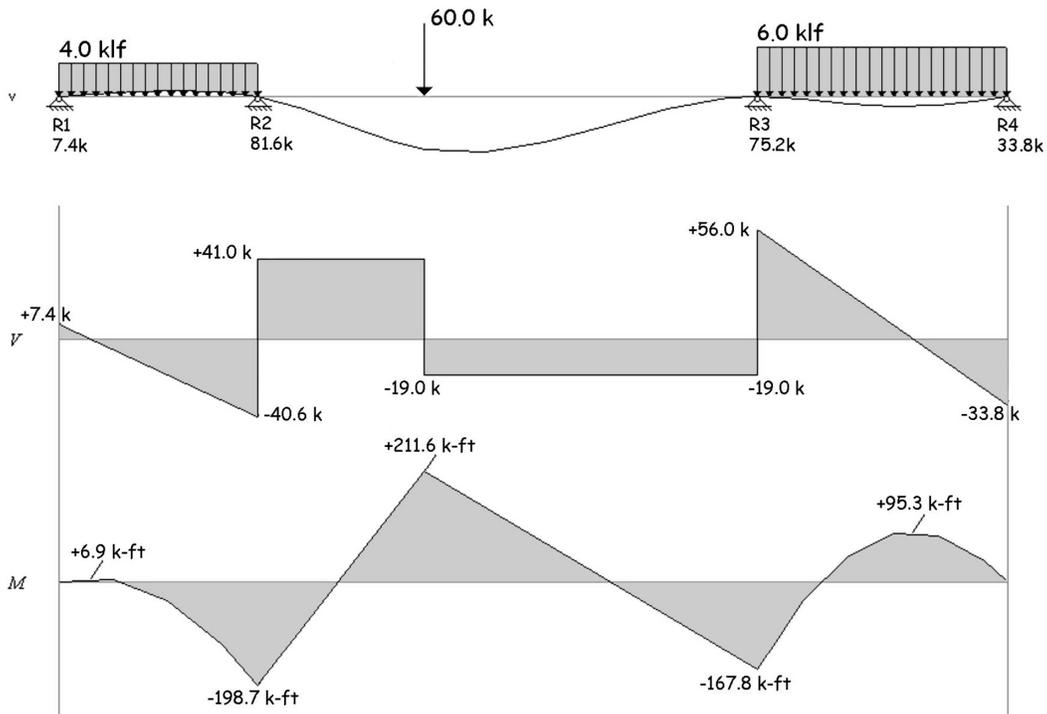
$$R_3 = 75.1531 \text{ K}$$

$$\sum M_{R_3} = -167.78 + 90(7.5) - R_4(15) = 0$$

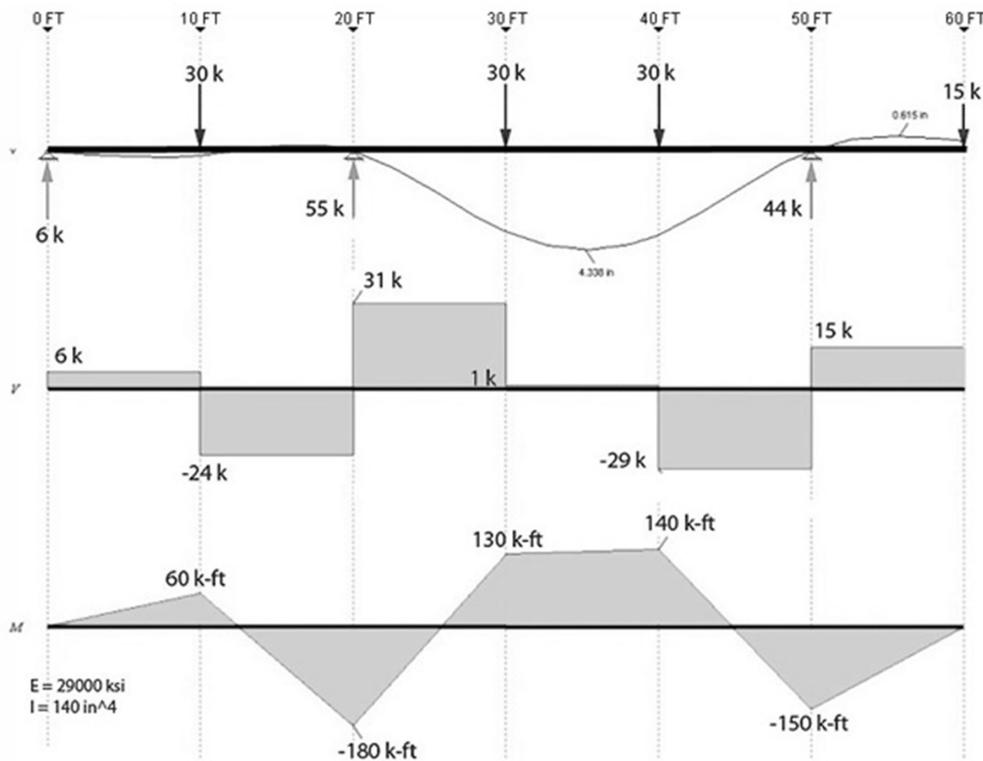
$$R_4 = 33.8147 \text{ K}$$

## Three-Moment Theorem Example (cont.)

9. Draw shear and moment diagrams as usual. This will also serve as a check for the moment values.

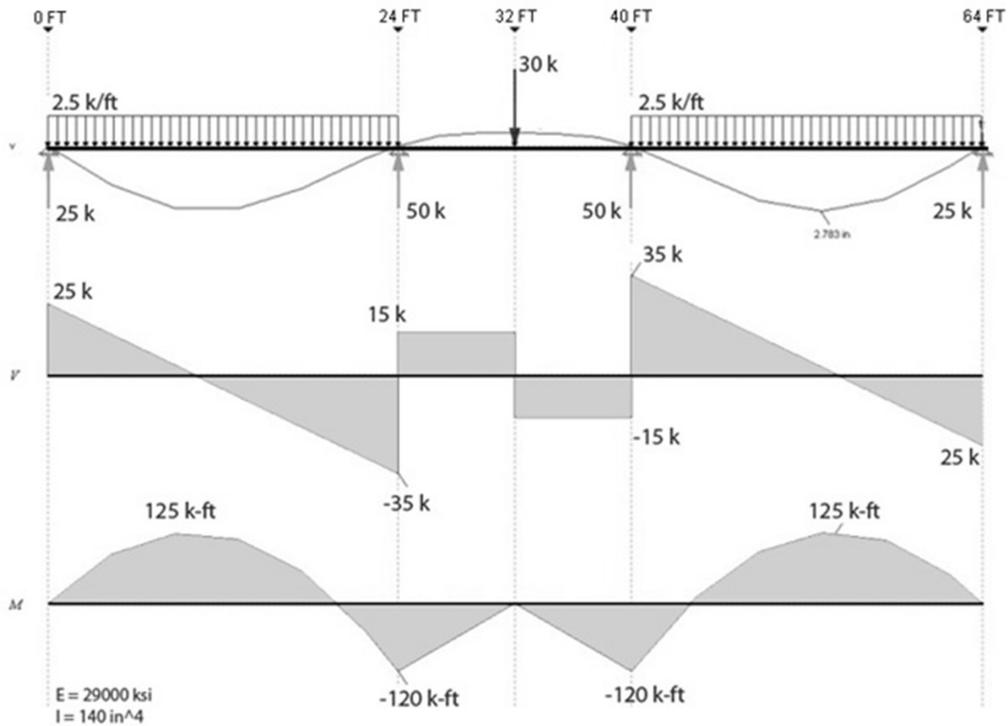


## Three-Moment Theorem – 2 Spans



$$M_A L_1 + 2M_B (L_1 + L_2) + M_C L_2 = 6[EI\Theta_1 + EI\Theta_2]$$

## Three-Moment Theorem – 3 Spans

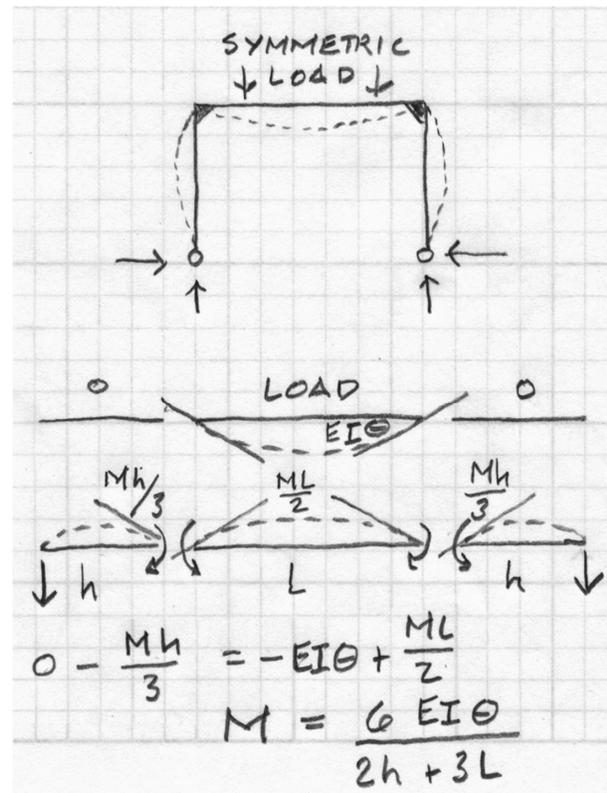


$$M_A L_1 + 2M_B (L_1 + L_2) + M_C L_2 = 6[EI\Theta_1 + EI\Theta_2]$$

## 2-Hinge Frame

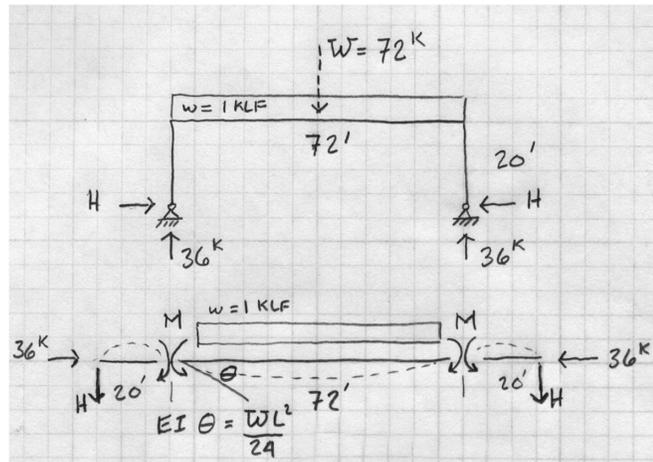
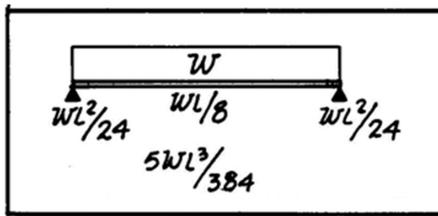
- Statically indeterminate
- Find negative moment at knee
- Sum of  $\Theta$ 's left and right of knee are equal
- Symmetric case solution is:

$$M = \frac{6 EI\Theta}{2h + 3L}$$



## 2-Hinge Frame example

- Symmetric case solution
- Vertical reactions by symmetry
- Find moment at knee
- With FBD of one leg find H

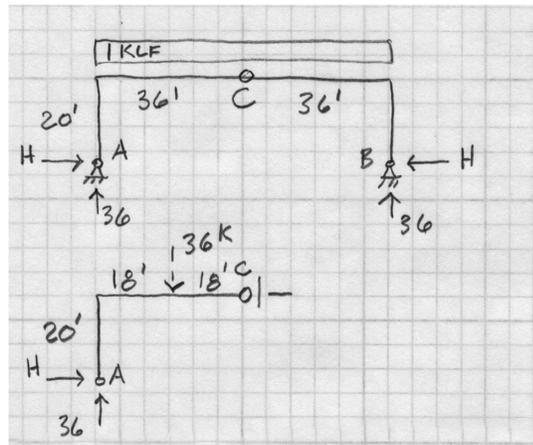


$$M = \frac{6 E I \theta}{2h + 3L} = \frac{6 \frac{72(72)^2}{24}}{2(20) + 3(72)} = \frac{364.5 \text{ K}^{-1}}{2(20) + 3(72)}$$

$$M = H(20), \quad H = \frac{M}{20} = \frac{364.5}{20} = 18.2 \text{ K}$$

## 3-Hinge Frame comparison

- Statically determinate
- Solve with statics
- FBD of half from hinge
- Solve for H
- Use FBD of leg to solve M



$$\sum M_C = 0 = -36(18) + 36(36) - H(20)$$

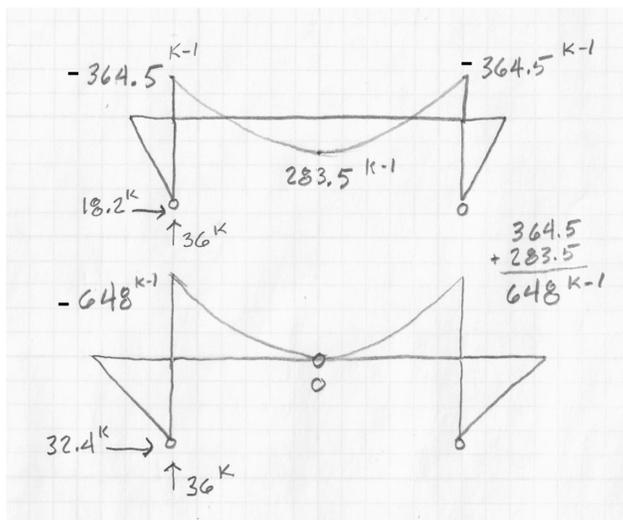
$$H = 32.4 \text{ K}$$

$$M = H(20) = 32.4(20) = 648 \text{ K}^{-1}$$

# Comparison of moments

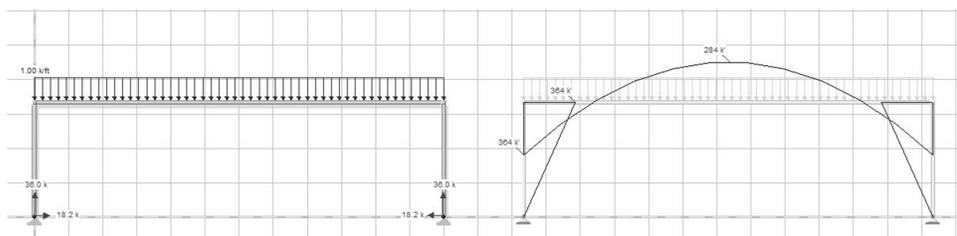
- 2-hinge frame

- 3-hinge frame



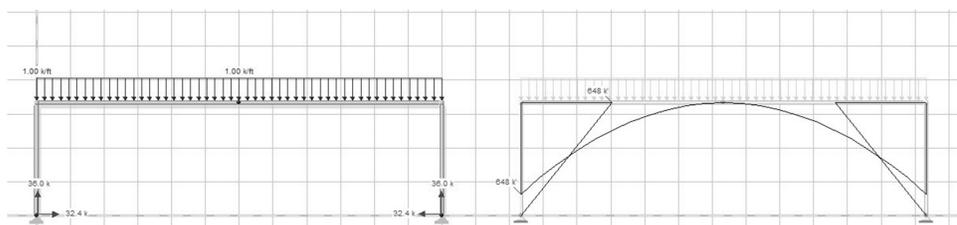
## The effect of shape and hinges

Moment:  
 knee:  $-364 \text{ ft-lbs}$   
 center:  $+284 \text{ ft-lbs}$   
 horz. react. =  $18.2 \text{ k}$



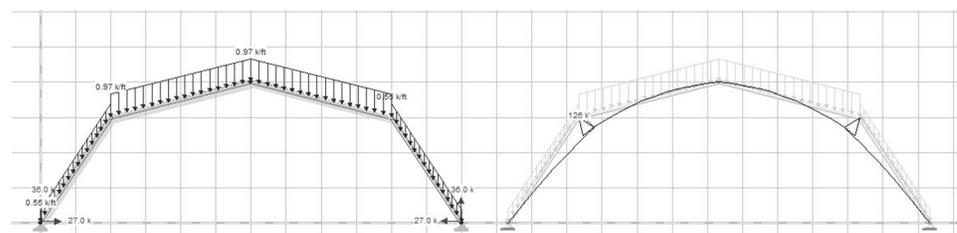
### Continuous Beam

Moment:  
 knee:  $-648 \text{ ft-lbs}$   
 center:  $0 \text{ ft-lbs}$   
 horz. react. =  $32.4 \text{ k}$



### Beam with center hinge

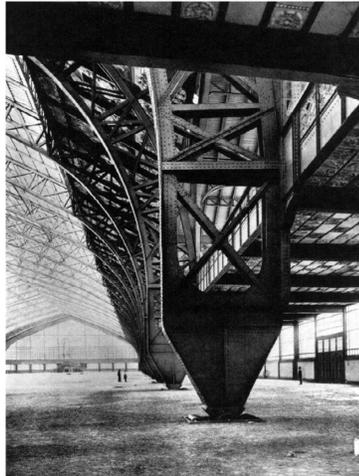
Moment:  
 knee:  $-126 \text{ ft-lbs}$   
 center:  $0 \text{ ft-lbs}$   
 horz. react. =  $27.0 \text{ k}$



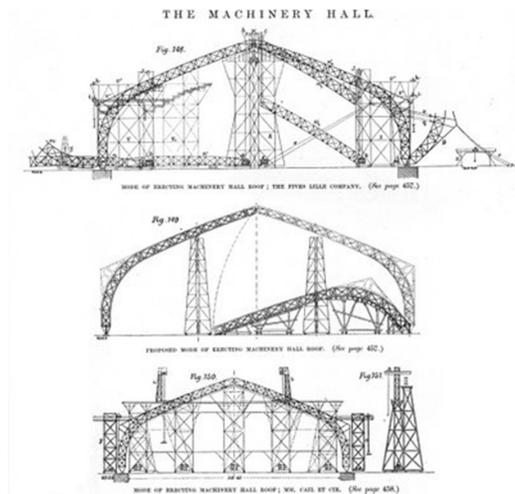
### 3 Hinged Arch

## Characteristics of a 3-Hinged Arch

- Statically determinate – can be calculated with statics
- Movement or settling of foundations will not alter member stresses
- Small fabrication errors in length do not affect internal stresses
- Hinge placement can reduce internal stresses

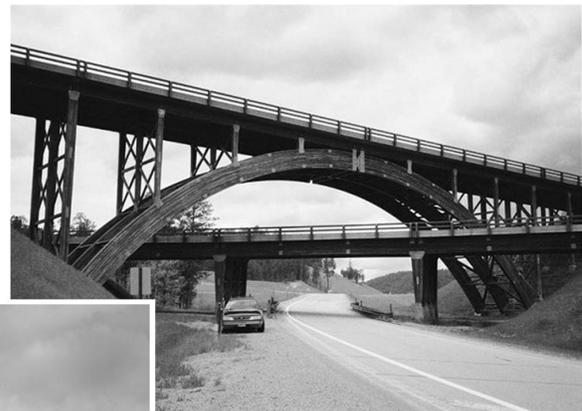


Gallery of the Machines, 1889 Paris  
 Architect: Ferdinand Dutert  
 Engineer: Victor Contamin



## Keystone Wye Bridge

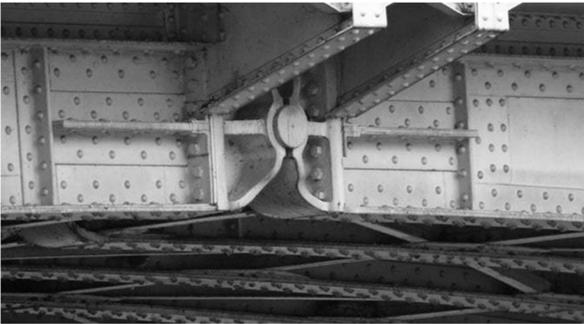
Glulam arches, spanning 160 ft.  
 built 1967-68 in South Dakota



# Examples and Details



Sydney Harbour Bridge



Sydney Harbour Bridge

Center Hinge



University of Michigan, TCAUP

Structures I

The Iron Bridge  
Telford England



Hinged Glulam Timbers