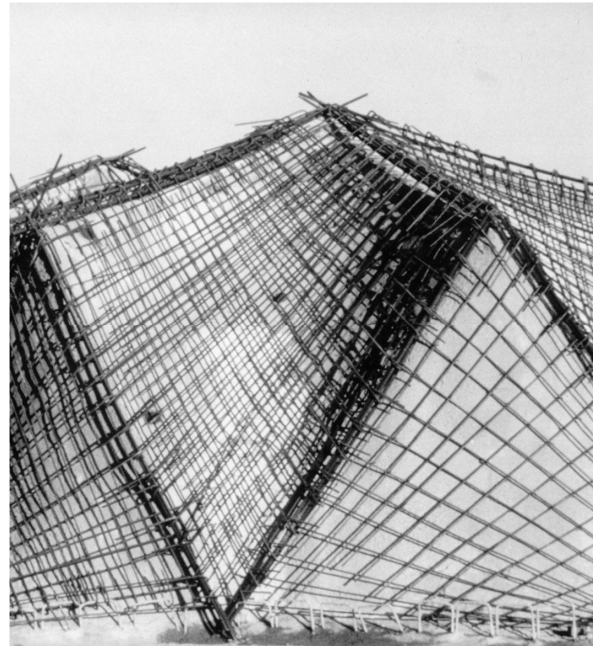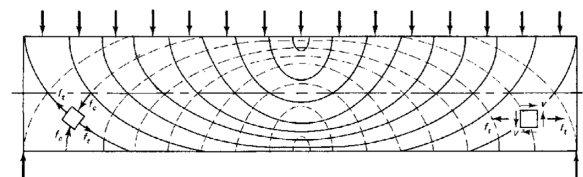# Reinforced Concrete Beams
# Ultimate Strength Design
(ACI 318-19) – PART I



- Flexure in Concrete
- Ultimate Strength Design (LRFD)
- Failure Modes
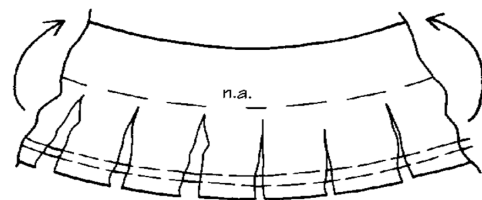- Flexure Equations
- Rectangular Beam Analysis

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# Flexure

The stress trajectories in this simple beam, show principal tension as solid lines.
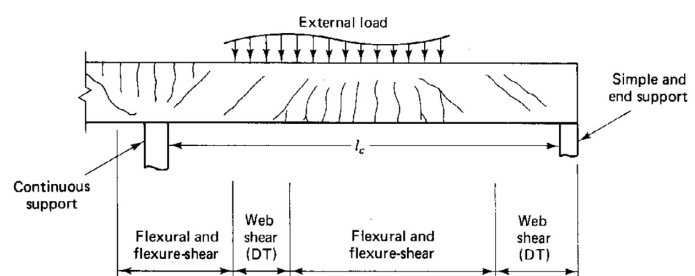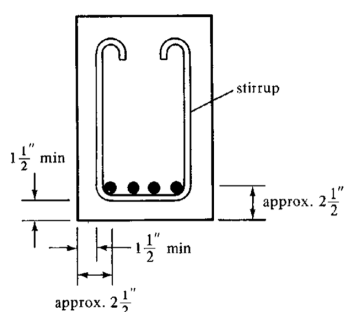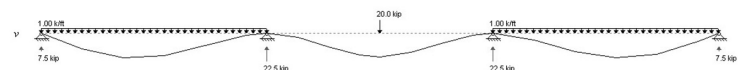
Reinforcement must be placed to resist these tensile forces

In beams continuous over supports, the stress reverses (negative moment).
In such areas, tensile steel is on top.

Shear reinforcement is provided by vertical or sloping stirrups.

# Ultimate Strength – (LRFD)

Nominal Strength ≥ Design Strength
(strength of member ≥ required by loads)

LRFD uses 2 safety factors: $\gamma$ and $\phi$
$\phi$ nominal strength ≥ $\gamma$ required strength

$\gamma$ increases the required strength of the
member and is placed on the loads

$\phi$ reduces the member strength capacity and is
placed on the calculated force

Loads increased:
$\gamma$ Factors: DL=1.2  LL=1.6
U is the required strength
U=1.2DL+1.6LL
(factors from ASCE 7)

Strength reduced:
$\phi$ Factors:  e.g. flexure = 0.9
in tension-controlled  beams

**Table 21.2.1—Strength reduction factors $\phi$**

| | Action or structural element | $\phi$ | Exceptions |
|---|---|---|---|
| (a) | Moment, axial force, or combined moment and axial force | 0.65 to 0.90 in accordance with 21.2.2 | Near ends of pretensioned members where strands are not fully developed, $\phi$ shall be in accordance with 21.2.3. |
| (b) | Shear | 0.75 | Additional requirements are given in 21.2.4 for structures designed to resist earthquake effects. |
| (c) | Torsion | 0.75 | — |
| (d) | Bearing | 0.65 | — |
| (e) | Post-tensioned anchorage zones | 0.85 | — |
| (f) | Brackets and corbels | 0.75 | — |
| (g) | Struts, ties, nodal zones, and bearing areas designed in accordance with strut-and-tie method in Chapter 23 | 0.75 | — |
| (h) | Components of connections of precast members controlled by yielding of steel elements in tension | 0.90 | — |
| (i) | Plain concrete elements | 0.60 | — |
| (j) | Anchors in concrete elements | 0.45 to 0.75 in accordance with Chapter 17 | — |

---

# Ultimate Strength – (ACI 318-14)

Reduced Nominal Strength ≥ Factored Load Effects

$$\Phi S_n \geq U$$

**$\gamma$ Factored Loads** (see ACSE 7)

1) 1.4D
2) 1.2D + 1.6L + 0.5(Lr or S or R)
3) 1.2D + 1.6(Lr or S or R) + (1.0L or 0.5W)
4) 1.2D + 1.0W + 1.0L + 0.5(Lr or S or R)
5) 1.2D + 1.0E + 1.0L + 0.2S
6) 0.9D + 1.0W
7) 0.9D + 1.0E

D = service dead loads
L = service live load
Lr = service roof live load
S = snow loads
W = wind loads
R = rainwater loads
E = earthquake loads

**Strength Reduction Factors, $\Phi$**

| | | |
|---|---|---|
| Mn | Flexural ($\varepsilon > 0.005$) | 0.90 |
| Vn | Shear | 0.75 |
| Pn | Compression (spiral) | 0.75 |
| Pn | Compression (other) | 0.65 |
| Bn | Bearing | 0.65 |
| Tn | Torsion | 0.75 |
| Nn | Tension | 0.90 |
| Combined stress | | 0.65 to 0.90 |

ACI 318 21.2.2
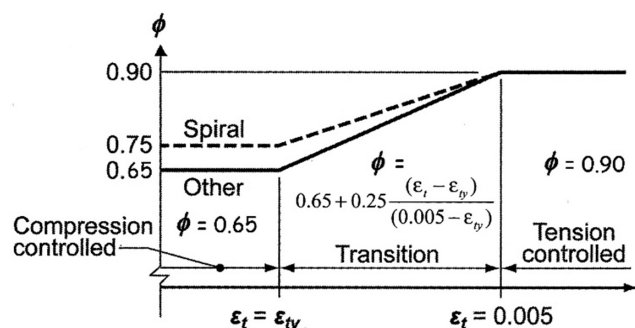
## Strength Measurement
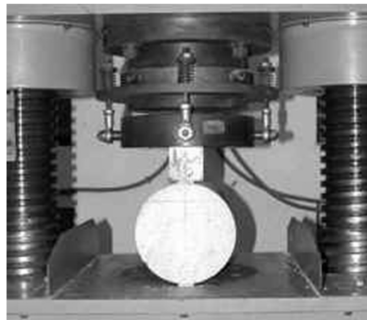
• Compressive strength
  – 12" x 6" cylinder
  – 28 day moist cure
  – Ultimate (failure) strength
  – Usable strain $\varepsilon_{cu}$ = 0.003 (ACI 318)

$$\boxed{f_c'}$$

• Tensile strength ASTM C496
  – 12" x 6" cylinder
  – 28 day moist cure
  – Ultimate (failure) strength
  – Split cylinder test
  – ca. 10% of f'c
  – Neglected in flexure analysis

$$\boxed{f_t'}$$

---

## Failure Modes Based on As



• **No Reinforcing**
  o Less than $As_{min}$
  o Brittle failure
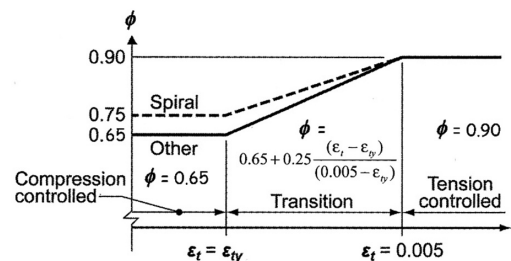
• **Reinforcing < balance (use this)**
  o Steel yields before concrete fails
  o Ductile failure
  o $\varepsilon_t \geq$ 0.005 for tension controlled

• **Reinforcing = balance**
  o Concrete fails just as steel yields
  o $r_{bal}$

• **Reinforcing > balance**
  o Concrete fails before steel yields
  o Low ductility
  o Sudden failure

As,min:
greater of a and b

$$\boxed{\rho = \frac{A_s}{bd}}$$

(a) $\dfrac{3\sqrt{f_c'}}{f_y} b_w d$

(b) $\dfrac{200}{f_y} b_w d$

$$\rho_{max} = 0.75\,\rho_b$$

$$\boxed{\rho_{bal} = \left(\frac{0.85\beta_1 f_c'}{f_y}\right)\left(\frac{87000}{87000 + f_y}\right)}$$
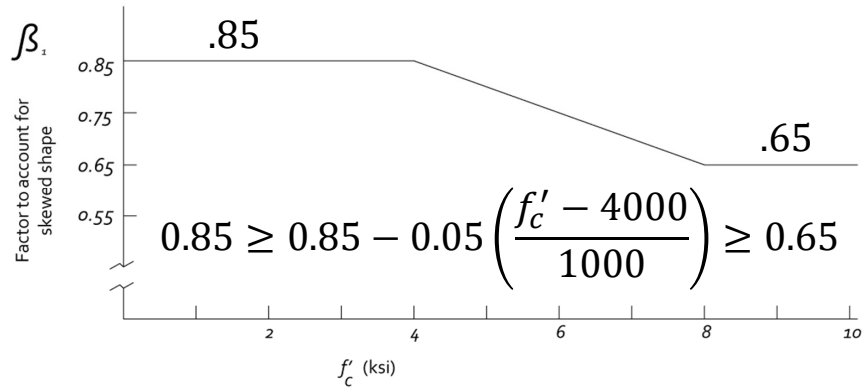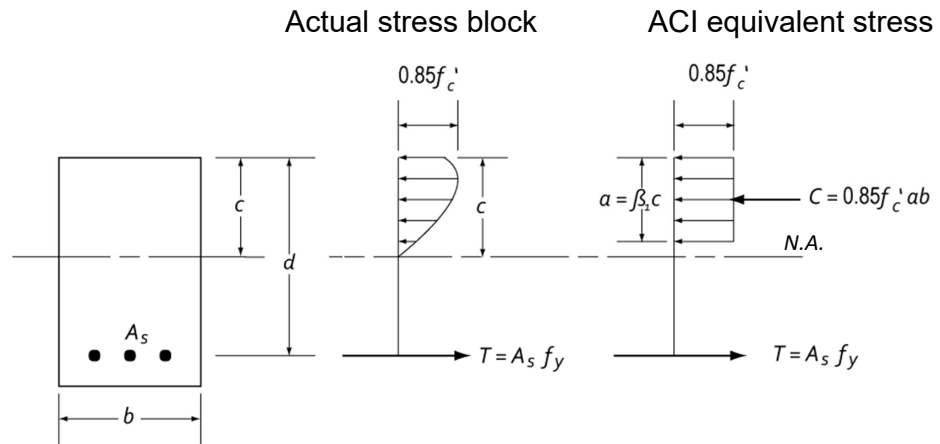
$$\boxed{As > As_{max} \qquad SuddenDeath!!}$$

## ACI Stress Block

$\beta_1$ is a factor to account for the non-linear shape of the compression stress block.
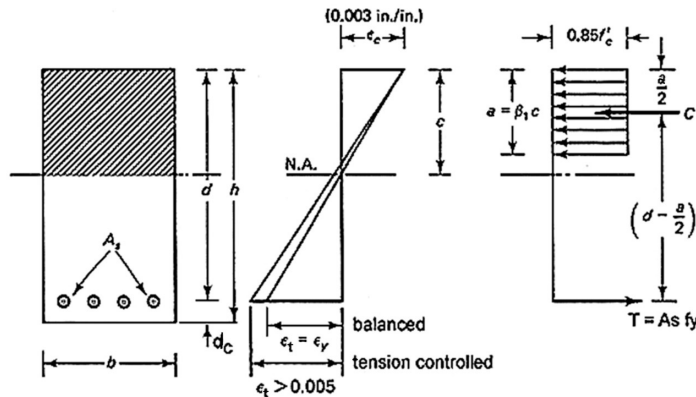
$$a = \beta_1 c$$

Actual stress block

ACI equivalent stress



$0.85f_c'$

$C = 0.85f_c'\,ab$

$a = \beta_1 c$

N.A.

$T = A_s\,f_y$

$T = A_s\,f_y$

### psi

| f'$_c$ | β$_1$ |
|-------|-------|
| 0 | 0.85 |
| 1000 | 0.85 |
| 2000 | 0.85 |
| 3000 | 0.85 |
| 4000 | 0.85 |
| 5000 | 0.8 |
| 6000 | 0.75 |
| 7000 | 0.7 |
| 8000 | 0.65 |
| 9000 | 0.65 |
| 10000 | 0.65 |



$\beta_1$

.85

.65

$$0.85 \geq 0.85 - 0.05\left(\frac{f_c' - 4000}{1000}\right) \geq 0.65$$

Factor to account for skewed shape

$f_c'$ (ksi)

---

## Flexure Equations

strain     ACI equivalent stress block



(0.003 in./in.)

$0.85f_c'$

$a = \beta_1 c$

N.A.

$\left(d - \frac{a}{2}\right)$

balanced
tension controlled

$\epsilon_t > 0.005$

$T = A_s\,f_y$

$$C = T$$

$$0.85f_c'ab = A_s f_y$$

solving for $a$,

$$a = \frac{A_s f_y}{0.85 f_c' b} = \frac{\rho f_y d}{0.85 f_c'}$$

$$\boxed{\varepsilon_t = \frac{d-c}{c}(0.003)} \qquad \boxed{\rho = \frac{A_s}{bd}}$$

$$M_n = T\left(d - \frac{a}{2}\right) = A_s f_y \left(d - \frac{a}{2}\right)$$

$$M_u = \phi M_n$$

$$M_u = \phi M_n = \phi A_s f_y \left(d - \frac{a}{2}\right)$$

$$M_u = \phi A_s f_y d \left(1 - 0.59 \frac{\rho f_y}{f_c'}\right)$$

# Balance Condition

From similar triangles at balance condition:

$$\frac{c}{d} = \frac{0.003}{0.003 + (f_y/E_s)} = \frac{0.003}{0.003 + (f_y/29 \times 10^6)}$$

$$c = \frac{87,000}{87,000 + f_y}d$$

Use equation for a. Substitute into $c = a / \beta_1$

$$a = \frac{\rho f_y d}{0.85 f_c'}$$
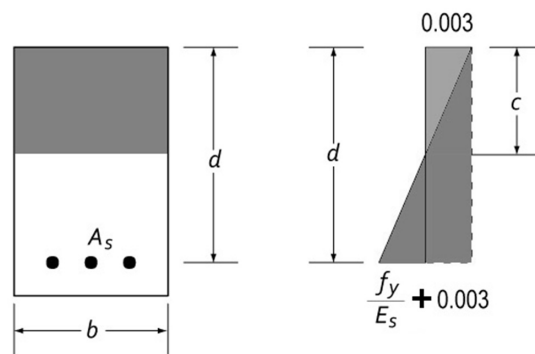
$$\boxed{\rho = \frac{A_s}{bd}}$$

$$c = \frac{a}{\beta_1} = \frac{\rho f_y d}{0.85 \beta_1 f_c'}$$

Equate expressions for c:

$$\frac{\rho f_y d}{0.85 \beta_1 f_c'} = \frac{87,000}{87,000 + f_y}d$$

$$\rho_b = \left(\frac{0.85 \beta_1 f_c'}{f_y}\right)\left(\frac{87,000}{87,000 + f_y}\right)$$



0.003

Strain diagram for balanced condition.

$$\frac{f_y}{E_s} + 0.003$$

**Table A.8** Balanced Ratio of Reinforcement $\rho_b$ for Rectangular Sections with Tension Reinforcement Only

| $f_c'$ | | 2,500 psi (17.2 MPa) $\beta_1 = 0.85$ | 3,000 psi (20.7 MPa) $\beta_1 = 0.85$ | 4,000 psi (27.6 MPa) $\beta_1 = 0.85$ | 5,000 psi (34.5 MPa) $\beta_1 = 0.80$ | 6,000 psi (41.4 MPa) $\beta_1 = 0.75$ |
|---|---|---|---|---|---|---|
| Grade 40 | $\rho_b$ | 0.0309 | 0.0371 | 0.0495 | 0.0582 | 0.0655 |
| 40,000 psi | $0.75\rho_b$ | 0.0232 | 0.0278 | 0.0371 | 0.0437 | 0.0492 |
| (275.8 MPa) | $0.50\rho_b$ | 0.0155 | 0.0186 | 0.0247 | 0.0291 | 0.0328 |
| Grade 50 | $\rho_b$ | 0.0229 | 0.0275 | 0.0367 | 0.0432 | 0.0486 |
| 50,000 psi | $0.75\rho_b$ | 0.0172 | 0.0206 | 0.0275 | 0.0324 | 0.0365 |
| (344.8 MPa) | $0.50\rho_b$ | 0.0115 | 0.0138 | 0.0184 | 0.0216 | 0.0243 |
| Grade 60 | $\rho_b$ | 0.0178 | 0.0214 | 0.0285 | 0.0335 | 0.0377 |
| 60,000 psi | $0.75\rho_b$ | 0.0134 | 0.0161 | 0.0214 | 0.0252 | 0.0283 |
| (413.7 MPa) | $0.50\rho_b$ | 0.0089 | 0.0107 | 0.0143 | 0.0168 | 0.0189 |
| Grade 75 | $\rho_b$ | 0.0129 | 0.0155 | 0.0207 | 0.0243 | 0.0274 |
| 75,000 psi | $0.75\rho_b$ | 0.0097 | 0.0116 | 0.0155 | 0.0182 | 0.0205 |
| (517.1 MPa) | $0.50\rho_b$ | 0.0065 | 0.0078 | 0.0104 | 0.0122 | 0.0137 |

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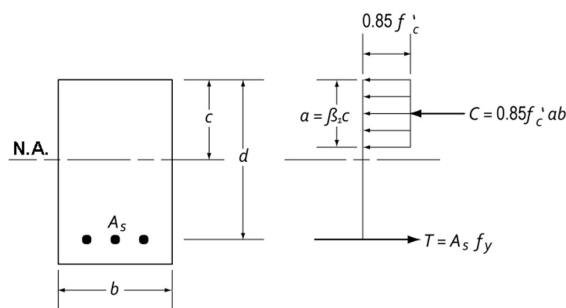# Rectangular Beam Analysis

Data:
- Section dimensions – b, h, (span)
- Steel area - As
- Material properties – f'c, fy

Required:
- Nominal Strength (of beam) Moment - Mn
- Required (by load) Design Moment – Mu
- Load capacity



0.85 f'c

$C = 0.85 f_c' ab$

$T = A_s f_y$

$As_{min}$:
greater of (a) and (b)

$$\boxed{c = \frac{a}{\beta_1}}$$

$$\boxed{\varepsilon_t = \frac{d - c}{c}0.003 \geq 0.005}$$

1. Calculate d
2. Check As min
3. Calculate a
4. Determine c
5. Check that $\varepsilon_t \geq 0.005$ (tension controlled)
6. Find nominal moment, Mn
7. Calculate required moment, $\phi$ Mn ≥ Mu

   (if $\varepsilon_t \geq 0.005$ then $\phi = 0.9$)
8. Determine max. loading (or span)

(a) $\dfrac{3\sqrt{f_c'}}{f_y}b_w d$

(b) $\dfrac{200}{f_y}b_w d$

$$\boxed{a = \frac{A_s f_y}{0.85 f_c' b}}$$

$$\boxed{M_n = A_s f_y \left(d - \frac{a}{2}\right)}$$

$$\boxed{\varphi M_n \geq M_u}$$

$$\boxed{M_u = \frac{(1.2 w_{DL} + 1.6 w_{LL})l^2}{8}}$$
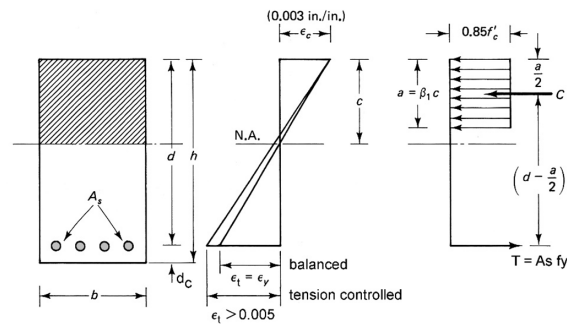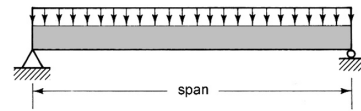
$$1.6 w_{LL} = \frac{M_u 8}{l^2} - 1.2 w_{DL}$$

# Rectangular Beam Analysis

**Data:**
- Dimensions – 12"x23"
- Steel – 4 x # 6    fy = 60ksi
- Concrete f'c = 6000 psi
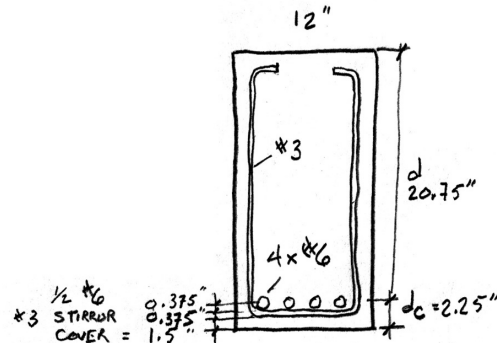- Stirrup # 3,  Cover 1.5"

**Required:**
- Required Moment – $\phi Mn = Mu$ (capacity)

1.  Calculate d

$$d_c = \text{COVER} + \#3 + \tfrac{1}{2}(\#6)$$
$$= 1.5 + 0.375 + \frac{0.75}{2} = 2.25"$$

$$d = h - d_c = 23" - 2.25 = 20.75"$$

12"

#3

$\frac{d}{20.75"}$

4 x #6

$\frac{1}{2}$ #6     0.375"
#3  STIRRUR   0.375"
COVER  =  1.5"

$d_c = 2.25"$

---

# Rectangular Beam Analysis cont.

Data:
dimensions – 12"x23"
Steel – 4 x # 6 – As = 1.76 in$^2$
f'c = 6000 psi  fy = 60ksi

**Table A.2** Designations, Areas, Perimeters, and Weights of Standard Bars

| | | Customary Units | | | SI Units | |
|---|---|---|---|---|---|---|
| Bar No. | Diameter (in.) | Cross-sectional Area (in.$^2$) | Unit Weight (lb/ft) | Diameter (mm) | Cross-sectional Area (mm$^2$) | Unit Weight (kg/m) |
| 3 | 0.375 | 0.11 | 0.376 | 9.52 | 71 | 0.560 |
| 4 | 0.500 | 0.20 | 0.668 | 12.70 | 129 | 0.994 |
| 5 | 0.625 | 0.31 | 1.043 | 15.88 | 200 | 1.552 |
| 6 | 0.750 | 0.44 | 1.502 | 19.05 | 284 | 2.235 |
| 7 | 0.875 | 0.60 | 2.044 | 22.22 | 387 | 3.042 |
| 8 | 1.000 | 0.79 | 2.670 | 25.40 | 510 | 3.973 |
| 9 | 1.128 | 1.00 | 3.400 | 28.65 | 645 | 5.060 |
| 10 | 1.270 | 1.27 | 4.303 | 32.26 | 819 | 6.404 |
| 11 | 1.410 | 1.56 | 5.313 | 35.81 | 1006 | 7.907 |
| 14 | 1.693 | 2.25 | 7.650 | 43.00 | 1452 | 11.384 |
| 18 | 2.257 | 4.00 | 13.600 | 57.33 | 2581 | 20.238 |

2.   Check As$_{min}$

As min

① $\frac{3\sqrt{f'_c}\, bd}{fy} = \frac{3\sqrt{6000}}{60000}(12 \times 20.75) = 0.964\, in^2$ ← CONTROLS

② $\frac{200\, bd}{fy} = \frac{200(12)(20.75)}{60000} = 0.83\, in^2$

∴ As min = 0.964 in$^2$

$As = A_b (N.B.) = 0.44(4) = 1.76\, in^2$  ✓ > 0.964 in$^2$

# Rectangular Beam Analysis cont.

Data:
dimensions – 12"x23"
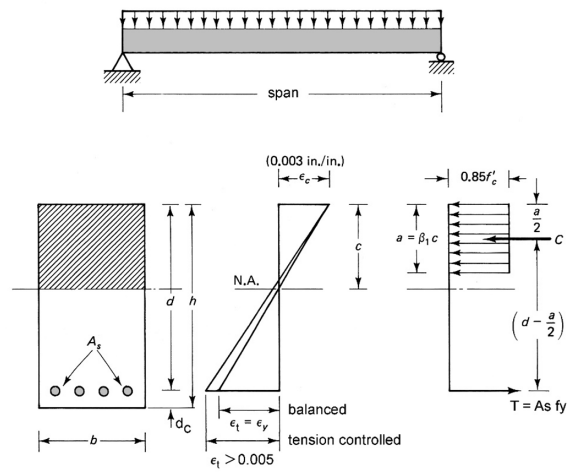Steel – 4 x # 6 – As = 1.76 in$^2$
f'c = 6000 psi  fy = 60ksi

Beta1:

$$0.85 \geq$$

$$0.85 - 0.05 \left( \frac{f_c' - 4000}{1000} \right)$$

$$\geq 0.65$$

| f'$_c$ | β$_1$ |
|--------|-------|
| 0 | 0.85 |
| 1000 | 0.85 |
| 2000 | 0.85 |
| 3000 | 0.85 |
| 4000 | 0.85 |
| 5000 | 0.8 |
| 6000 | 0.75 |
| 7000 | 0.7 |
| 8000 | 0.65 |
| 9000 | 0.65 |
| 10000 | 0.65 |

3. Find a

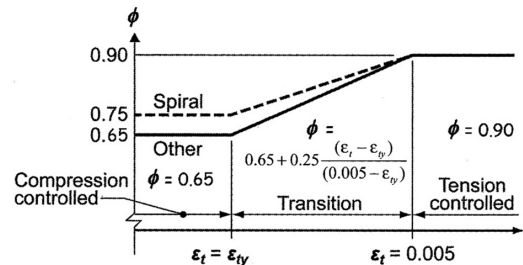$$a = \frac{As\,fy}{0.85\,f_c'\,b} = \frac{(1.76)(60)}{.85(6)(12)} = 1.725''$$

4. Find c

$$\beta_1 = 0.85 - 0.05 \frac{f_c' - 4000}{1000} = 0.85 - 0.1 = 0.75$$

$$c = \frac{a}{\beta_1} = \frac{1.725}{0.75} = 2.300''$$

---

# Rectangular Beam Analysis cont.

$$\boxed{\varepsilon_t = \frac{d - c}{c}\, 0.003 \geq 0.005}$$

5. Check that $\varepsilon_t \geq 0.005$
   (for tension controlled section)
   $\phi = 0.9$

$$\varepsilon_t = \frac{d-c}{c}\, 0.003 = \frac{20.75 - 2.3}{2.3}\, 0.003$$

$$= 0.02406 > 0.004 \therefore ok \checkmark$$

$$= 0.02406 > 0.005 \therefore \text{tension controlled}$$

6. Find nominal moment, Mn

$$T = A_s\, f_y = 1.76^{in^2}(60^{KSI}) = 105.6\ K$$

$$Mn = T\left(d - \frac{a}{2}\right) = 105.6\left(20.75 - \frac{1.725''}{2}\right)$$

$$Mn = 2100\ K\text{-}''$$

7. Calculate required moment
   $\phi Mn \geq Mu$

$$\phi Mn = 0.9(2100) = 1890\ K\text{-}''$$

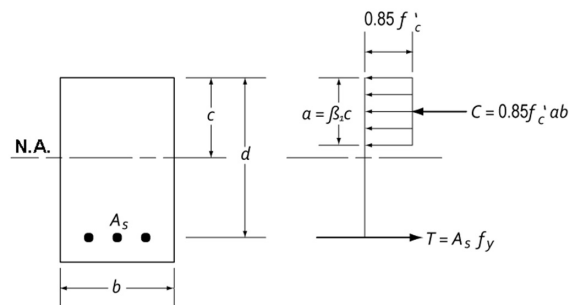$$M_U = \phi M_n = 1890/12 = \underline{157.5}\ K\text{-}'$$

# One-way Slab Analysis

Data:
- Section dimensions – b, h, (span)
- Steel area – As , bar diam. $b_d$ , o.c. spacing
- Material properties – f'c,  fy

Required:
- Nominal Strength (of beam) Moment - Mn
- Required (by load) Design Moment – Mu
- Load capacity



1. Calculate d = h – cover – $bar_d$/2
2. Find As/ft. Check As min
3. Calculate a
4. Determine c
5. Check that $\varepsilon_t \geq 0.005$ (tension controlled)
6. Find nominal moment, Mn
7. Calculate required moment,

   $\phi$ Mn ≥ Mu (if $\varepsilon_t \geq 0.005$ then $\phi$ = 0.9)
8. Determine max. loading (or span)

$$c = \frac{a}{\beta_1}$$

$$\varepsilon_t = \frac{d - c}{c} 0.003 \geq 0.005$$

**Table 7.6.1.1—$A_{s,min}$ for nonprestressed one-way slabs**

| Reinforcement type | $f_y$, psi | | $A_{s,min}$ |
|---|---|---|---|
| Deformed bars | < 60,000 | | $0.0020A_g$ |
| Deformed bars or welded wire reinforcement | ≥ 60,000 | Greater of: | $\frac{0.0018 \times 60,000}{f_y} A_g$ |
| | | | $0.0014A_g$ |

$$a = \frac{A_s f_y}{0.85 f_c' b}$$

$$M_n = A_s f_y \left( d - \frac{a}{2} \right)$$

$$\varphi M_n \geq M_u$$

As/ft = As x 12/o.c.

Ag =  b h

$$M_u = \frac{(1.2 w_{DL} + 1.6 w_{LL}) l^2}{8}$$

$$1.6 w_{LL} = \frac{M_u 8}{l^2} - 1.2 w_{DL}$$

---

# Slab Analysis

Data:
- Span = 18 ft
- h = 11"
  take b = 12"
- Steel  #8 @ 18" o.c.
- f'$_c$ = 3000 psi
- f$_y$ = 60 ksi

| Bar size designation | Nominal cross section area, sq. in. | Weight, lb per ft | Nominal diameter, in. |
|---|---|---|---|
| #3 | 0.11 | 0.376 | 0.375 |
| #4 | 0.20 | 0.668 | 0.500 |
| #5 | 0.31 | 1.043 | 0.625 |
| #6 | 0.44 | 1.502 | 0.750 |
| #7 | 0.60 | 2.044 | 0.875 |
| #8 | 0.79 | 2.670 | 1.000 |
| #9 | 1.00 | 3.400 | 1.128 |
| #10 | 1.27 | 4.303 | 1.270 |
| #11 | 1.56 | 5.313 | 1.410 |
| #14 | 2.25 | 7.650 | 1.693 |
| #18 | 4.00 | 13.600 | 2.257 |



wu = 1.2 DL + 1.6 LL

18'

Required:
- Design moment capacity – M$_u$
- Maximum LL in PSF

$$d = 11" - \frac{1}{2} - \frac{3}{4} = 9.75"$$

$$A_s = \frac{12"}{18"} (0.79 in^2)$$
$$= 0.5267 in^2/FT$$

1. Find d
2. Find A$_s$
   Check A$_{s,min}$

   Ag = 12" x 11" = 132 in$^2$

   [0.0018(60)/60] 132 = 0.237 in$^2$

   0.0014 (132) = 0.1848 in$^2$

   0.527 > 0.237  ok
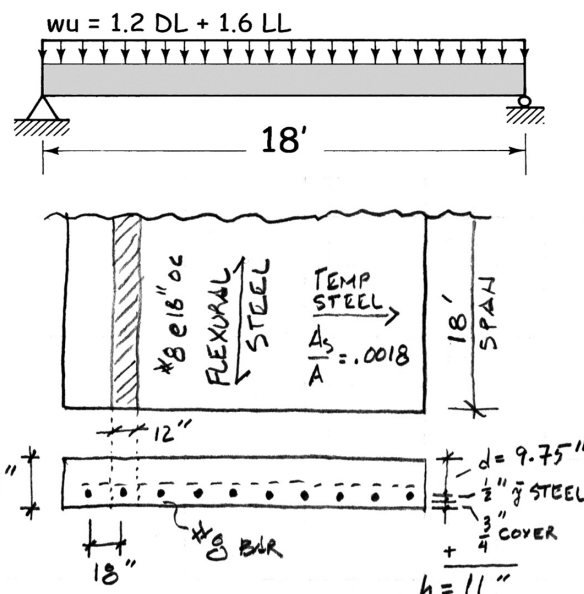
**Table 7.6.1.1—$A_{s,min}$ for nonprestressed one-way slabs**

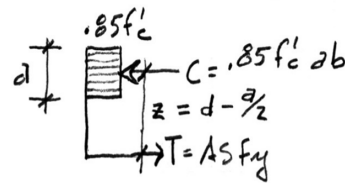| Reinforcement type | $f_y$, psi | | $A_{s,min}$ |
|---|---|---|---|
| Deformed bars | < 60,000 | | $0.0020A_g$ |
| Deformed bars or welded wire reinforcement | ≥ 60,000 | Greater of: | $\frac{0.0018 \times 60,000}{f_y} A_g$ |
| | | | $0.0014A_g$ |

ACI 318-14

# Slab Analysis

| $f'_c$ | $\beta_1$ |
|------|------|
| 0 | 0.85 |
| 1000 | 0.85 |
| 2000 | 0.85 |
| 3000 | 0.85 |
| 4000 | 0.85 |
| 5000 | 0.8 |
| 6000 | 0.75 |
| 7000 | 0.7 |
| 8000 | 0.65 |
| 9000 | 0.65 |
| 10000 | 0.65 |

3. Find a

$$d = \frac{A_s f_y}{.85 f'_c b} = \frac{0.5267(60)}{.85(3)(12)} = 1.033"$$



$$C = .85 f'_c \, ab$$
$$z = d - \tfrac{a}{2}$$
$$T = A_s f_y$$

4. Find c = $\beta_1$ a

$$C = \frac{a}{\beta_1} = \frac{1.033"}{0.85} = 1.215"$$

5. Check failure mode
   $\varepsilon_t \geq$ 0.005 for tension controlled

$$\varepsilon_t = \frac{0.003\, d}{c} - 0.003$$

$$\varepsilon_t = \frac{0.003\,(9.75")}{1.215"} - 0.003 = 0.021\, \text{"/\text{"}}$$

6. Find force T

$$\varepsilon_t = 0.021 > 0.005 \therefore \text{TENSION CONTROLLED}$$
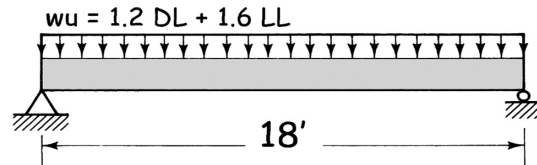
7. Find moment arm z

8. Find nominal strength moment, $M_n$

$$T = A_s f_y = 0.5267(60) = 31.6^K$$
$$z = d - \tfrac{a}{2} = 9.75 - \frac{1.033}{2} = 9.23"$$

$$M_n = Tz = 31.6(9.23) = 291.8^{K-"}$$

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# Slab Analysis



wu = 1.2 DL + 1.6 LL

18'

9. Find required moment, $M_u$

$$M_u = \phi M_n = 0.9(291.8)\frac{1000}{12} = 21885^{'-\#}$$

10. Find slab DL

$$w_{DL} = \gamma_c \frac{h}{12} = 150 \frac{11}{12} = 137.5 \text{ PSF}$$

11. Determine max. loading

$$M_u = 21885^{'-\#} = \frac{(1.2\, w_{DL} + 1.6\, w_{LL})\,\ell^2}{8}$$

$$\frac{21885(8)}{(18')^2} = 1.2(137.5) + 1.6(w_{LL})$$

$$540.37 = 165 + 1.6(w_{LL})$$

$$w_{LL} = 234.6 \text{ PSF}$$