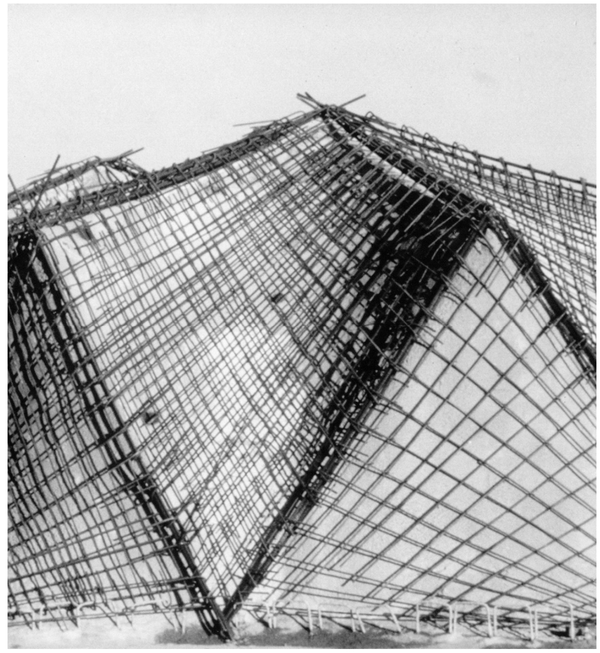
Architecture 324
Structures II

Reinforced Concrete Beams Ultimate Strength Design

(ACI 318-19) – PART I



- Flexure in Concrete
- Ultimate Strength Design (LRFD)
- Failure Modes
- Flexure Equations
- Rectangular Beam Analysis

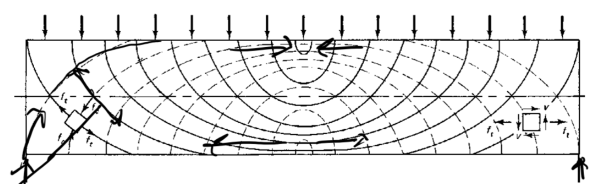
Flexure

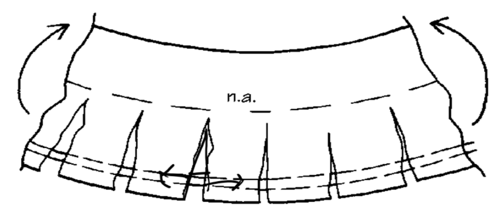
The stress trajectories in this simple beam, show principal tension as solid lines.

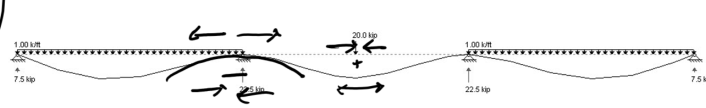
Reinforcement must be placed to resist these tensile forces

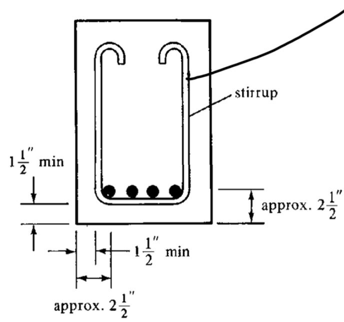
In beams continuous over supports, the stress reverses (negative moment).
In such areas, tensile steel is on top.

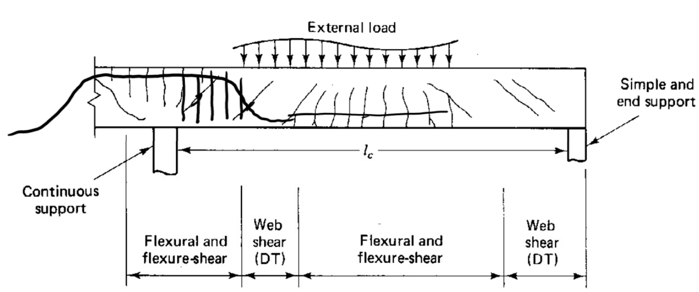
Shear reinforcement is provided by vertical or sloping stirrups.











Ultimate Strength – (LRFD)

Nominal Strength ≥ Design Strength
(strength of member ≥ required by loads)

LRFD uses 2 safety factors: γ and ϕ
ϕ nominal strength ≥ γ required strength

γ increases the required strength of the member and is placed on the loads

ϕ reduces the member strength capacity and is placed on the calculated force

Loads increased:
γ Factors: DL=1.2 LL=1.6
U is the required strength
U=1.2DL+1.6LL
(factors from ASCE 7)

Strength reduced:
ϕ Factors: e.g. flexure = 0.9
in tension-controlled beams

Table 21.2.1—Strength reduction factors ϕ

	Action or structural element	ϕ	Exceptions
(a)	Moment, axial force, or combined moment and axial force	0.65 to 0.90 in accordance with 21.2.2	Near ends of pretensioned members where strands are not fully developed, ϕ shall be in accordance with 21.2.3.
(b)	Shear	0.75	Additional requirements are given in 21.2.4 for structures designed to resist earthquake effects.
(c)	Torsion	0.75	—
(d)	Bearing	0.65	—
(e)	Post-tensioned anchorage zones	0.85	—
(f)	Brackets and corbels	0.75	—
(g)	Struts, ties, nodal zones, and bearing areas designed in accordance with strut-and-tie method in Chapter 23	0.75	—
(h)	Components of connections of precast members controlled by yielding of steel elements in tension	0.90	—
(i)	Plain concrete elements	0.60	—
(j)	Anchors in concrete elements	0.45 to 0.75 in accordance with Chapter 17	—

Ultimate Strength – (ACI 318-14)

Reduced Nominal Strength ≥ Factored Load Effects

ΦSn ≥ U

γ Factored Loads (see ACSE 7)

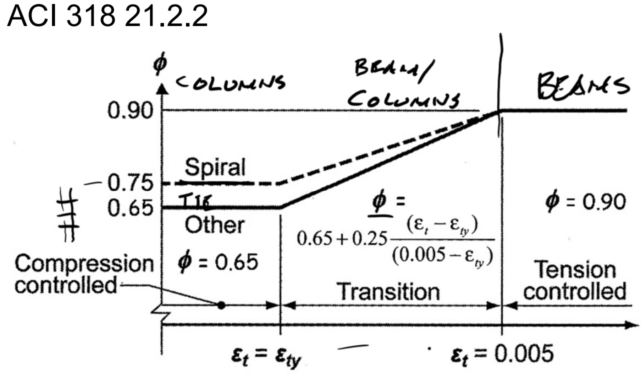
1) 1.4D
2) 1.2D + 1.6L + 0.5(Lr or S or R)
3) 1.2D + 1.6(Lr or S or R) + (1.0L or 0.5W)
4) 1.2D + 1.0W + 1.0L + 0.5(Lr or S or R)
5) 1.2D + 1.0E + 1.0L + 0.2S
6) 0.9D + 1.0W
7) 0.9D + 1.0E

D = service dead loads
L = service live load
Lr = service roof live load
S = snow loads
W = wind loads
R = rainwater loads
E = earthquake loads

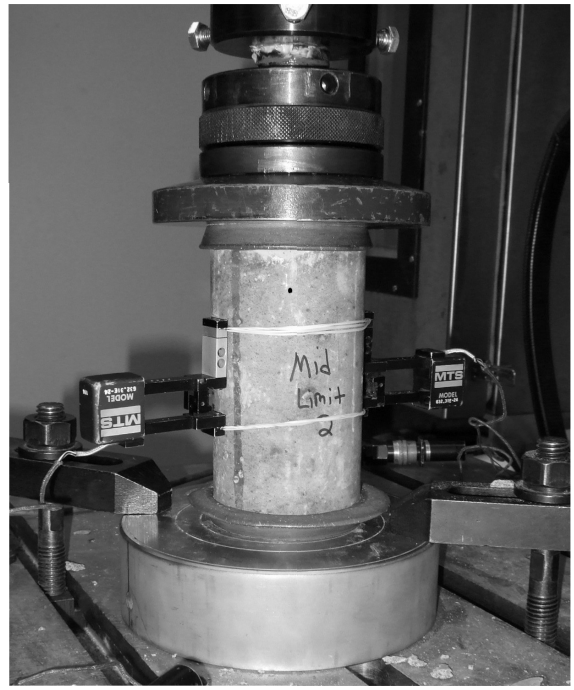
Strength Reduction Factors, Φ

Mn	Flexural (ε > 0.005)	0.90
Vn	Shear	0.75
Pn	Compression (spiral)	0.75
Pn	Compression (other)	0.65
Bn	Bearing	0.65
Tn	Torsion	0.75
Nn	Tension	0.90
Combined stress		0.65 to 0.90

ACI 318 21.2.2

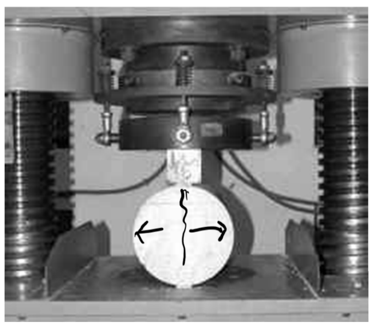


Strength Measurement

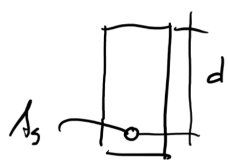


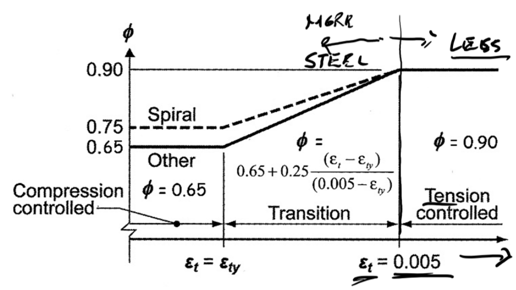
- Compressive strength f_c'
  - 12" x 6" cylinder
  - 28 day moist cure
  - Ultimate (failure) strength
  - Usable strain ε_{cu} = 0.003 (ACI 318)

- Tensile strength ASTM C496 f_t'
  - 12" x 6" cylinder
  - 28 day moist cure
  - Ultimate (failure) strength
  - Split cylinder test
  - ca. 10% of f'c
  - Neglected in flexure analysis



Failure Modes Based on As





- **No Reinforcing**
  - Less than As_{min}
  - Brittle failure
- **Reinforcing < balance (use this)**
  - Steel yields before concrete fails
  - Ductile failure
  - $\varepsilon_t \geq$ 0.005 for tension controlled
- **Reinforcing = balance**
  - Concrete fails just as steel yields
  - ρ_{bal}
- **Reinforcing > balance**
  - Concrete fails before steel yields
  - Low ductility
  - Sudden failure

$$\rho = \frac{A_s}{bd}$$

As,min:
greater of a and b

(a) $\frac{3\sqrt{f_c'}}{f_y} b_w d$

(b) $\frac{200}{f_y} b_w d$

$$\rho_{\mathbf{max}} = 0.75\,\rho_b$$

$$\rho_{bal} = \left(\frac{0.85\beta_1 f_c'}{f_y}\right)\left(\frac{87000}{87000 + f_y}\right)$$

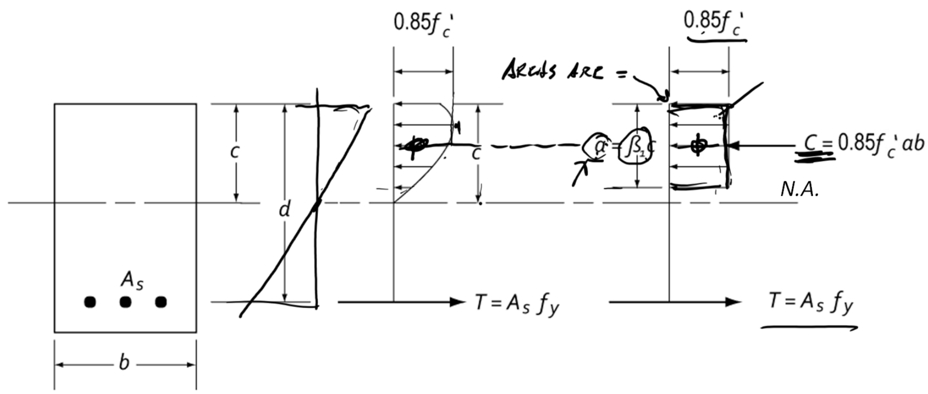
As > As_{max} SuddenDeath!!

ACI Stress Block

Actual stress block ACI equivalent stress

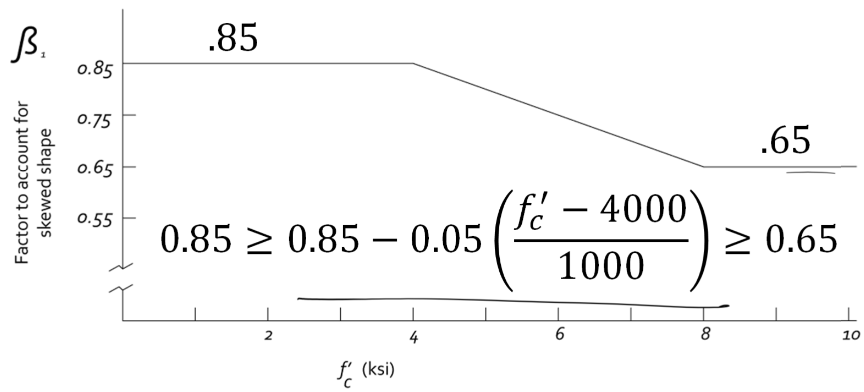
β_1 is a factor to account for the non-linear shape of the compression stress block.

$$\boxed{a = \beta_1 c}$$



psi

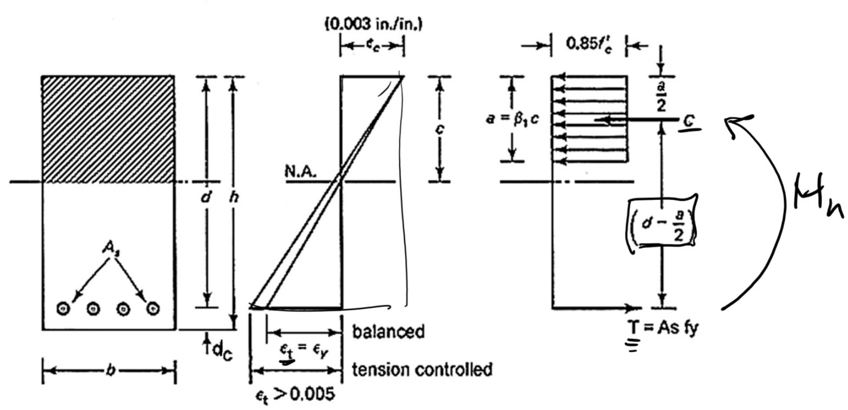
f'c	β_1
0	0.85
1000	0.85
2000	0.85
3000	0.85
4000	0.85
5000	0.8
6000	0.75
7000	0.7
8000	0.65
9000	0.65
10000	0.65



$$0.85 \geq 0.85 - 0.05\left(\frac{f_c' - 4000}{1000}\right) \geq 0.65$$

Flexure Equations

strain ACI equivalent stress block



$$C = T$$

$$0.85 f_c' a b = A_s f_y$$

solving for a,

$$a = \frac{A_s f_y}{0.85 f_c' b} = \frac{\rho f_y d}{0.85 f_c'}$$

$\frac{A_s}{bd}$

$$M_n = T\left(d - \frac{a}{2}\right) = A_s f_y\left(d - \frac{a}{2}\right)$$

$$M_u = \phi M_n$$

$$M_u = \phi M_n = \phi A_s f_y\left(d - \frac{a}{2}\right)$$

$$M_u = \phi A_s f_y d\left(1 - 0.59\frac{\rho f_y}{f_c'}\right)$$

$$\boxed{\varepsilon_t = \frac{d - c}{c}(0.003)}$$

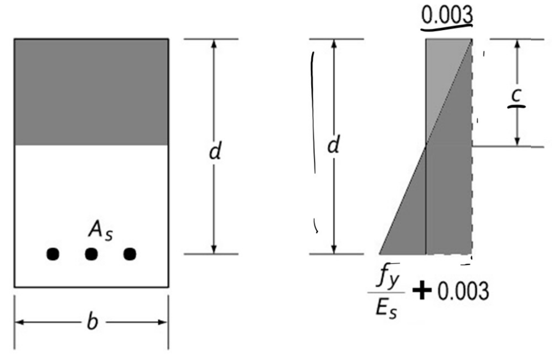
$$\boxed{\rho = \frac{A_s}{bd}}$$

Balance Condition

From similar triangles at balance condition:

$$\frac{c}{d} = \frac{0.003}{0.003 + (f_y/E_s)} = \frac{0.003}{0.003 + (f_y/29 \times 10^6)}$$

$$c = \frac{87{,}000}{87{,}000 + f_y} d$$



Strain diagram for balanced condition.

Use equation for a. Substitute into c = a / β_1

$$a = \frac{\rho f_y d}{0.85 f_c'}$$

$$\boxed{\rho = \frac{A_s}{bd}}$$

$$c = \frac{a}{\beta_1} = \frac{\rho f_y d}{0.85 \beta_1 f_c'}$$

Equate expressions for c:

$$\frac{\rho f_y d}{0.85 \beta_1 f_c'} = \frac{87{,}000}{87{,}000 + f_y} d$$

$$\rho_b = \left(\frac{0.85 \beta_1 f_c'}{f_y}\right)\left(\frac{87{,}000}{87{,}000 + f_y}\right)$$

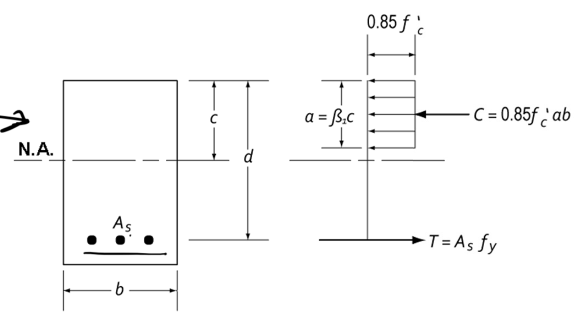
Table A.8 Balanced Ratio of Reinforcement ρ_b for Rectangular Sections with Tension Reinforcement Only

f_y	f_c'	2,500 psi (17.2 MPa) $\beta_1 = 0.85$	3,000 psi (20.7 MPa) $\beta_1 = 0.85$	4,000 psi (27.6 MPa) $\beta_1 = 0.85$	5,000 psi (34.5 MPa) $\beta_1 = 0.80$	6,000 psi (41.4 MPa) $\beta_1 = 0.75$
Grade 40	ρ_b	0.0309	0.0371	0.0495	0.0582	0.0655
40,000 psi	$0.75\rho_b$	0.0232	0.0278	0.0371	0.0437	0.0492
(275.8 MPa)	$0.50\rho_b$	0.0155	0.0186	0.0247	0.0291	0.0328
Grade 50	ρ_b	0.0229	0.0275	0.0367	0.0432	0.0486
50,000 psi	$0.75\rho_b$	0.0172	0.0206	0.0275	0.0324	0.0365
(344.8 MPa)	$0.50\rho_b$	0.0115	0.0138	0.0184	0.0216	0.0243
Grade 60	ρ_b	0.0178	0.0214	0.0285	0.0335	0.0377
60,000 psi	$0.75\rho_b$	0.0134	0.0161	0.0214	0.0252	0.0283
(413.7 MPa)	$0.50\rho_b$	0.0089	0.0107	0.0143	0.0168	0.0189
Grade 75	ρ_b	0.0129	0.0155	0.0207	0.0243	0.0274
75,000 psi	$0.75\rho_b$	0.0097	0.0116	0.0155	0.0182	0.0205
(517.1 MPa)	$0.50\rho_b$	0.0065	0.0078	0.0104	0.0122	0.0137

Rectangular Beam Analysis

Data:

- Section dimensions – b, h, (span)
- Steel area - As
- Material properties – f'c, fy



Required:

- Nominal Strength (of beam) Moment - Mn
- Required (by load) Design Moment – Mu
- Load capacity

As_{min}: greater of (a) and (b)

(a) $\frac{3\sqrt{f_c'}}{f_y} b_w d$

(b) $\frac{200}{f_y} b_w d$

$$\boxed{c = \frac{a}{\beta_1}}$$

$$\boxed{\varepsilon_t = \frac{d - c}{c} 0.003 \geq 0.005}$$

$$\boxed{a = \frac{A_s f_y}{0.85 f_c' b}}$$

$$\boxed{M_n = A_s f_y \left(d - \frac{a}{2}\right)}$$

$$\boxed{\varphi M_n \geq M_u}$$

Load

$$M_u = \frac{(1.2 w_{DL} + 1.6 w_{LL}) l^2}{8}$$

$$1.6 w_{LL} = \frac{M_u 8}{l^2} - 1.2 w_{DL}$$

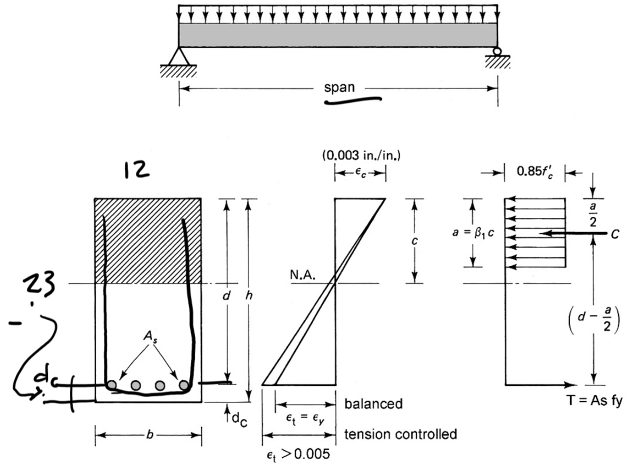
1. Calculate d
2. Check As min
3. Calculate a
4. Determine c
5. Check that $\varepsilon_t \geq 0.005$ (tension controlled)
6. Find nominal moment, Mn
7. Calculate required moment, ϕ Mn ≥ Mu

   (if $\varepsilon_t \geq 0.005$ then $\phi = 0.9$)
8. Determine max. loading (or span)

Rectangular Beam Analysis

Data:
- Dimensions – 12”x23”
- Steel – 4 x # 6   fy = 60ksi
- Concrete f’c = 6000 psi
- Stirrup # 3, Cover 1.5”

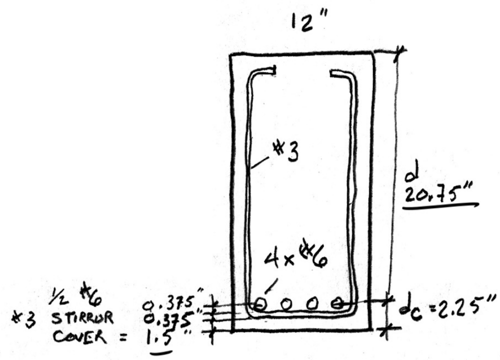
Required:
- Required Moment – ϕMn = Mu (capacity)



1. Calculate d

$$d_c = \text{COVER} + \#3 + \tfrac{1}{2}(\#6)$$
$$= 1.5 + 0.375 + \frac{0.75}{2} = 2.25''$$
STIRRUP

$$d = h - d_c = 23'' - 2.25 = \underline{20.75''}$$



Rectangular Beam Analysis cont.

Data:
dimensions – 12”x23”
Steel – 4 x # 6 – As = 1.76 in^2
f’c = 6000 psi fy = 60ksi

Table A.2 Designations, Areas, Perimeters, and Weights of Standard Bars

	Customary Units			SI Units		
Bar No.	Diameter (in.)	Cross-sectional Area (in.^{2})	Unit Weight (lb/ft)	Diameter (mm)	Cross-sectional Area (mm^2)	Unit Weight (kg/m)
3	0.375	0.11	0.376	9.52	71	0.560
4	0.500	0.20	0.668	12.70	129	0.994
5	0.625	0.31	1.043	15.88	200	1.552
6	0.750	0.44	1.502	19.05	284	2.235
7	0.875	0.60	2.044	22.22	387	3.042
8	1.000	0.79	2.670	25.40	510	3.973
9	1.128	1.00	3.400	28.65	645	5.060
10	1.270	1.27	4.303	32.26	819	6.404
11	1.410	1.56	5.313	35.81	1006	7.907
14	1.693	2.25	7.650	43.00	1452	11.384
18	2.257	4.00	13.600	57.33	2581	20.238

2. Check As$_{min}$ ✓

As min   CONC.   LARGER

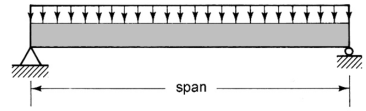
① $\frac{3\sqrt{f'_c}\, bd}{f_y}$ STEEL $= \frac{3\sqrt{6000}}{60{,}000}(12 \times 20.75) = \underline{0.964 \text{ in}^2}$ ← CONTROLS

② $\frac{200\, bd}{f_y} = \frac{200(12)(20.75)}{60000} = 0.83 \text{ in}^2$

$\therefore A_{s\,min} = \underline{0.964 \text{ in}^2}$

$\underline{A_s} = A_b\,(\text{N.B.}) = \underset{A_{s\,bar}}{0.44}(4) = \underline{1.76 \text{ in}^2}$ ✓ $> 0.964 \text{ in}^2$

Rectangular Beam Analysis cont.

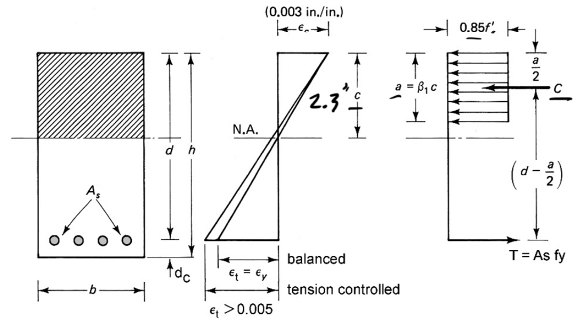


Data:
dimensions – 12”x23”
Steel – 4 x # 6 – As = 1.76 in^2
f’c = 6000 psi fy = 60ksi

Beta1:

$$0.85 \geq 0.85 - 0.05\left(\frac{f_c' - 4000}{1000}\right) \geq 0.65$$

f'_c	β_1
0	0.85
1000	0.85
2000	0.85
3000	0.85
4000	0.85
5000	0.8
6000	0.75
7000	0.7
8000	0.65
9000	0.65
10000	0.65



3. Find a

$$a = \frac{A_s f_y}{0.85 f_c' b} = \frac{(1.76)(60)}{.85(6)(12)} = 1.725''$$

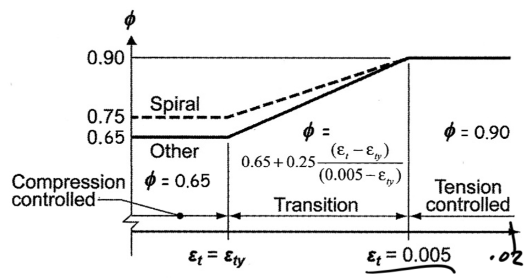
4. Find c

$$\beta_1 = 0.85 - 0.05\frac{f_c' - 4000}{1000} = 0.85 - 0.1 = 0.75$$

$$c = \frac{a}{\beta_1} = \frac{1.725}{0.75} = 2.300''$$

Rectangular Beam Analysis cont.

$$\boxed{\varepsilon_t = \frac{d-c}{c}0.003 \geq 0.005}$$



5. Check that $\varepsilon_t \geq 0.005$
(for tension controlled section)
ϕ = 0.9

$$\varepsilon_t = \frac{d-c}{c}0.003 = \frac{20.75 - 2.3}{2.3}0.003$$

$$= 0.02406 > 0.004 \therefore ok$$

$$= 0.02406 > 0.005 \therefore \text{tension controlled}$$

6. Find nominal moment, Mn

$$T = A_s f_y = 1.76^{in^2}(60^{KSI}) = 105.6 \text{ K}$$

$$M_n = T\left(d - \frac{a}{2}\right) = 105.6\left(20.75 - \frac{1.725''}{2}\right)$$

$$M_n = 2100 \text{ K-''}$$

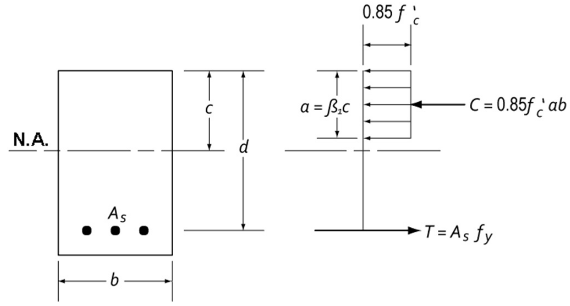
7. Calculate required moment
ϕMn ≥ Mu

$$\phi M_n = 0.9(2100) = 1890 \text{ K-''}$$

$$M_U = \phi M_n = 1890/12 = 157.5 \text{ K-'}$$

One-way Slab Analysis

Data:
- Section dimensions – b, h, (span)
- Steel area – As , bar diam. b_d , o.c. spacing
- Material properties – f'c, fy



Required:
- Nominal Strength (of beam) Moment - Mn
- Required (by load) Design Moment – Mu
- Load capacity

$$c = \frac{a}{\beta_1}$$

$$\varepsilon_t = \frac{d-c}{c}0.003 \geq 0.005$$

1. Calculate d = h – cover – $bar_d/2$
2. Find As/ft. Check As min
3. Calculate a
4. Determine c
5. Check that $\varepsilon_t \geq 0.005$ (tension controlled)
6. Find nominal moment, Mn
7. Calculate required moment,

   ϕ Mn ≥ Mu (if $\varepsilon_t \geq 0.005$ then ϕ = 0.9)
8. Determine max. loading (or span)

Table 7.6.1.1—$A_{s,min}$ for nonprestressed one-way slabs

Reinforcement type	f_y, psi		$A_{s,min}$
Deformed bars	< 60,000		$0.0020A_g$
Deformed bars or welded wire reinforcement	≥ 60,000	Greater of:	$\frac{0.0018\times 60{,}000}{f_y}A_g$
			$0.0014A_g$

$$a = \frac{A_s f_y}{0.85 f'_c b}$$

$$M_n = A_s f_y\left(d - \frac{a}{2}\right)$$

$$\varphi M_n \geq M_u$$

As/ft = As x 12/o.c.

Ag = b h

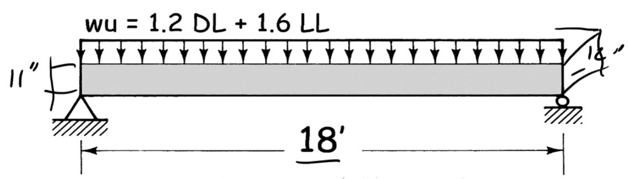
$$M_u = \frac{(1.2w_{DL} + 1.6w_{LL})l^2}{8}$$

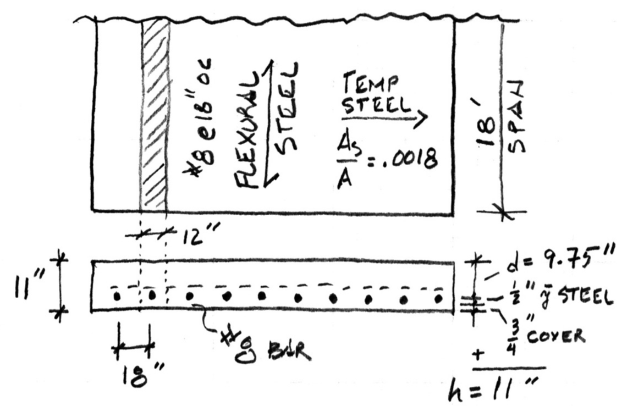
$$1.6w_{LL} = \frac{M_u 8}{l^2} - 1.2w_{DL}$$

Slab Analysis

Data:
- Span = 18 ft
- h = 11"
  take b = 12"
- Steel #8 @ 18" o.c.
- f'_c = 3000 psi
- f_y = 60 ksi

Bar size designation	Nominal cross section area, sq. in.	Weight, lb per ft	Nominal diameter, in.
#3	0.11	0.376	0.375
#4	0.20	0.668	0.500
#5	0.31	1.043	0.625
#6	0.44	1.502	0.750
#7	0.60	2.044	0.875
#8	0.79	2.670	1.000
#9	1.00	3.400	1.128
#10	1.27	4.303	1.270
#11	1.56	5.313	1.410
#14	2.25	7.650	1.693
#18	4.00	13.600	2.257





Required:
- Design moment capacity – M_u
- Maximum LL in PSF

$d = 11" - \frac{1}{2} - \frac{3}{4} = 9.75"$

$A_s = \frac{12"}{18"}(0.79\,in^2)$ #8

$= 0.5267\,in^2/FT$

1. Find d
2. Find A_s
   Check $A_{s,min}$

   Ag = 12" x 11" = 132 in^2

   [0.0018(60)/60] 132 = 0.237 in^2

   0.0014 (132) = 0.1848 in^2

   0.527 > 0.237 ok

Table 7.6.1.1—$A_{s,min}$ for nonprestressed one-way slabs

Reinforcement type	f_y, psi		$A_{s,min}$
Deformed bars	< 60,000		$0.0020A_g$
Deformed bars or welded wire reinforcement	≥ 60,000	Greater of:	$\frac{0.0018\times 60{,}000}{f_y}A_g$
			$0.0014A_g$

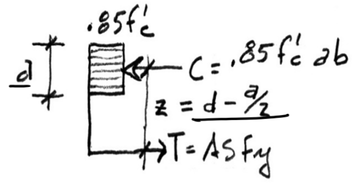
ACI 318-14

Slab Analysis

f'_c	β_1
0	0.85
1000	0.85
2000	0.85
$\boxed{3000}$	0.85
4000	0.85
5000	0.8
6000	0.75
7000	0.7
8000	0.65
9000	0.65
10000	0.65

3. Find a

$$a = \frac{A_s f_y}{.85 f'_c b} = \frac{0.5267(60)}{.85(3)(12)} = 1.033''$$



4. Find c = β_1 a

$$c = \frac{a}{\beta_1} = \frac{1.033''}{0.85} = 1.215''$$

5. Check failure mode
   $\varepsilon_t \geq$ 0.005 for tension controlled

$$\varepsilon_t = \frac{0.003\,d}{c} - 0.003$$

$$\varepsilon_t = \frac{0.003(9.75'')}{1.215''} - 0.003 = 0.021 \text{ "/"}$$

$$\varepsilon_t = 0.021 > 0.005 \therefore \text{TENSION CONTROLLED}$$

6. Find force T

$$T = A_s f_y = 0.5267(60) = 31.6^K$$

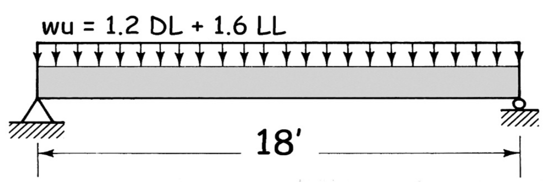
7. Find moment arm z

$$z = d - \frac{a}{2} = 9.75 - \frac{1.033}{2} = 9.23''$$

8. Find nominal strength moment, M_n

$$M_n = Tz = 31.6(9.23) = 291.8^{K\text{-}''}$$

Slab Analysis



9. Find required moment, M_u

$$M_u = \phi M_n = 0.9(291.8)\frac{1000}{12} = 21885^{\prime\text{-}\#}$$

10. Find slab DL

$$w_{DL} = \gamma_c \frac{h}{12} = \underset{\text{PCF}}{150}\,\frac{11}{12} = 137.5 \text{ PSF}$$

11. Determine max. loading

$$M_u = 21885^{\prime\text{-}\#} = \frac{(1.2\,w_{DL} + 1.6\,\overset{?}{w_{LL}})\,\ell^2}{8}$$

$$\frac{21885(8)}{(18')^2} = 1.2(137.5) + 1.6(w_{LL})$$

$$540.37 = 165 + 1.6(w_{LL})$$

$$w_{LL} = 234.6 \text{ PSF}$$